Discrete Mathematics Comprehensive Exam
Fall 2025

Student Number: ☐

Instructions: Complete **exactly 5** of the given 6 problems and **circle** their numbers below. The uncircled problems will **not** be graded.

1 2 3 4 5 6

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

All graphs in the following problems are finite and simple.

1. Let G be a cubic graph. Show that $V(G)$ has a bipartition X, Y such that $|X| = |Y|$ and $e(X, Y) \geqslant 2e(G)/3$, where $e(G)$ is the number of edges in G and $e(X, Y)$ is the number of edges of G between X and Y.

2. Let G be a 2-connected cubic plane graph. Show that if G is 4-face-colorable then G is also 3-edge-colorable. (Hint: Consider a 4-face-coloring using elements from the cyclic group $\mathbb{Z}_4$.) Is this still true if G is not 2-connected? Explain your answer.

3. Let $k \geqslant 2$ be an integer and G be a connected graph.
   (1) Show that if $|V(G)| \geqslant k + 1$ and $\delta(G) \geqslant k/2$ then G contains a path of length k. (This implies $ex(n, P_k) = \frac{k-1}{2}n$.)
   (2) Show that if G has no path of length k then $\chi'(G) \leqslant (k + 1)/2$.

4. Define a real matrix M to be n-*rich* if every $n \times n$ submatrix of M has rank at least 2 (over $\mathbb{R}$). Prove that for all n sufficiently large, if $K = 2^{0.4n}$ then there exists an n-rich 0-1 matrix (i.e. all entries are 0 or 1) M of dimensions $K \times K$.

5. Prove that every nonempty set S of positive integers has a subset T of size at least $\frac{1}{5}|S|$ such that there are no solutions to $a + 3b = c$ for $a, b, c \in T$.

6. Let $k \geqslant 3$ and let G be a graph where every vertex in G lies in at most d cycles of length k. Prove that if $d \leqslant k^{-1}2^{k-3}$, then there exists a 2-coloring of the vertices of G so that no k-cycle is monochromatic.