Differential Equations Comprehensive Exam
Fall 2025

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. How many periodic orbits does $\dot{x} = -\sin(x)$ with $x(t) \in \mathbb{R}$ admit? How about $\ddot{x} = -\sin(x)$? Fixed points do not count, and time translation does not contribute to a new orbit either.

2. What is the solution to $\ddot{x} + x + \sin(\omega t) = 0$ with $x(0) = 1, \dot{x}(0) = 0$?

3. Given constant $a \neq 0$, does the solution to $\dot{x} = a(x^4 - 4x^3 + 6x^2 - 4x + 1)$ exist for all $t > 0$? Why / why not?

4. Consider the following planar, autonomous system:

$$\begin{cases} \dot{x} &= -x + y^2 \\ \dot{y} &= -2x^2 + 2xy^2 \end{cases}$$

   Prove that $y = x^2$ is an invariant manifold of this vector field. Prove that there exists a heteroclinic connection between the equilibrium points (0,0) and (1,1).

5. Assume that $f(x) \in C^1(\mathbf{R})$ is uniformly bounded function with bounded and continuous first order derivatives. Consider the following initial value problem

$$\begin{cases} u_t + uu_x = -3u, \ x \in \mathbf{R}, t > 0, \\ u(x,0) = f(x). \end{cases}$$

   Prove that, this initial value problem admits a unique global (that is for all $x \in \mathbf{R}$ and $t \geq 0$) C^1 solution $u(x,t)$ if and only if $f'(x) \geq -3$ for all $x \in \mathbf{R}$.

6. Let Ω be an open bounded subset in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. If $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ is a solution of the following boundary value problem

$$\begin{cases} -\Delta u = -u^5, \ x \in \Omega, \\ u(x) = 0, \ for \ x \in \partial\Omega, \end{cases}$$

   Prove that $u(x)$ must be identically zero on Ω.

7. For any finite constant $c \in \mathbf{R}$, prove that there is at most one solution $u \in C^2([0,1] \times [0,\infty))$ to the following problem

$$\begin{cases} u_{tt} - u_{xx} + cu_t + u = f(x,t), \ x \in (0,1), \ t > 0, \\ u(0,t) = u(1,t) = 0, \ t > 0 \\ u(x,0) = g(x), \ u_t(x,0) = h(x), \ for \ x \in (0,1), \end{cases}$$

8. Consider the following Porous Medium equation

$$\rho_t = (\rho^m)_{xx},$$

where $\rho(x,t) \geq 0$, and $m \geq 1$. Like the heat equation, Porous Medium equation admits self-similar solutions in the form of

$$\rho(x,t) = At^{\alpha}\phi(\frac{x}{t^{\beta}})$$

for some constants α, β, and A, in sense that the equation for ϕ is an ODE in terms of the new variable $\xi = \frac{x}{t^{\beta}}$.

Determine α and β for the following two classes of self-similar solutions

1) $\lim_{x\to-\infty} \rho(x,t) = \rho_- > 0$ and $\lim_{x\to-\infty} \rho(x,t) = \rho_+ > 0$

2) $\lim_{x\to-\infty} \rho(x,t) = 0$ and $\displaystyle\int_{-\infty}^{+\infty} \rho(x,t)\, dx = M > 0.$