Algebra Comprehensive Exam
Fall 2025

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let G be a finite group of order 231 . Prove that G has a normal subgroup of order 77.

2. Let G be a group of order p^4 where p is prime. Suppose the center of G has order p^2. Find the number of conjugacy classes of G. [**Hint:** Use the fact that the center of G is contained in the centralizer of every element of G.]

3. Let R be a commutative ring with 1. An element $a \in R$ is called *nilpotent* if there is a positive integer k such that $a^k = 0$. Prove that the set of all nilpotent elements of R forms an ideal.

4. Let R be a commutative Noetherian ring with 1.

   (a) Show that every surjective homomorphism $f : R \to R$ is an isomorphism.

   (b) Give a concrete counterexample to show that the statement is false without the assumption on being Noetherian.

5. Prove that $\mathbf{Q}(\sqrt{2}+\sqrt{3}) = \mathbf{Q}(\sqrt{2},\sqrt{3})$.

6. Let $A \in \mathrm{Mat}_{n\times n}(F)$ be an idempotent matrix, i.e., $A^2 = A$, with entries in a field F. Prove that $\mathrm{Im}(A) \oplus \ker(A) = F^n$.

7. Let $T : V \to V$ be a linear operator on a finite-dimensional vector space over a field F. Show that
$$\mathrm{rank}(T) + \mathrm{rank}(T^3) \geq 2\,\mathrm{rank}(T^2).$$

8. Let $R = \mathbb{Z}[i]$ be the ring of Gaussian integers, where i is a root of x^2+1. Let $f : R^2 \to R^2$ be the map given by $f(x, y) = (ix + y, x + iy)$, and let N be the image of f. Find $a \in \mathbf{Z}$ such that $R^2/N \cong R/aR$.