Probability Comprehensive Exam
Spring 2024

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $X = (X_1, \ldots, X_4)$ be a mean zero normal random vector in $\mathbb{R}^4$ with covariance matrix $\Sigma = (\sigma_{ij} : 1 \leq i, j \leq 4)$. Show that
$$\mathbb{E}X_1X_2X_3X_4 = \sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}.$$

2. Let $X_1, \ldots, X_n, \ldots$ be i.i.d. exponential random variables with expectation 1 and let $Y_1, \ldots, Y_n, \ldots$ be i.i.d. standard normal random variables. Show that
$$\max(X_1, \ldots, X_n) - \max(Y_1, \ldots, Y_n) \to \infty \text{ as } n \to \infty \text{ a.s.}$$

3. Suppose $X_0 = 0$ and
$$X_n = aX_{n-1} + \xi_n, n \geq 1,$$
where $a \in \mathbb{R}$ and $\xi_1, \xi_2, \ldots$ are independent mean zero random variables. Let $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$ be the σ-field generated by $X_1, \ldots, X_n$. Show that
$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = aX_n.$$

4. Let $f : [0, 1] \mapsto \mathbb{R}$ be a twice continuously differentiable function. Show that
$$\lim_{n\to\infty} n \int_0^1 \cdots \int_0^1 \Big[f\Big(\frac{x_1 + \ldots x_n}{n}\Big) - f(1/2)\Big] dx_1 \ldots dx_n = \frac{f''(1/2)}{24}.$$

5. (i) Let X be a non-zero random variable and let $b_1, b_2 \in (0, +\infty)$ be such that $b_1X \stackrel{d}{=} b_2X$ where d indicates equality in distribution. Show that $b_1 = b_2$.

   (ii) Let X be a non-constant random variable and let $b_1, b_2 \in (0, +\infty)$ and $c_1, c_2 \in \mathbb{R}$ be such that $b_1X + c_1 \stackrel{d}{=} b_2X + c_2$ where again d indicates equality in distribution. Show that $b_1 = b_2$ and $c_1 = c_2$.

6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Two random variables X et Y are said to be independent conditionally to $\mathcal{G}$ if for any non-negative measurable functions, f and g, one has:
$$\mathbb{E}[f(X)g(Y)\,|\mathcal{G}] = \mathbb{E}[f(X)\,|\mathcal{G}]\ \mathbb{E}[g(Y)\,|\mathcal{G}]. \qquad (C_1)$$

   (i) What is the meaning of (C_1) if $\mathcal{G} = \{\emptyset, \Omega\}$? And if $\mathcal{G} = \mathcal{F}$?

   (ii) Show that the definition (C_1) is equivalent to the fact that for any non-negative random variable Z which is $\mathcal{G}$-mesurable, and for any non-negative measurable functions f and g one has
$$\mathbb{E}[f(X)g(Y)Z] = \mathbb{E}\big[f(X)Z\mathbb{E}[g(Y)|\mathcal{G}]\big]. \qquad (C_2)$$

7. Let D be a domain of $\mathbb{R}^d$ and let f be a measurable real-valued function defined on $\mathbb{R}^d$ such that $\mathbf{1}_D f$ is Lebesgue integrable. Let $(U_n)_{n\geq 1}$ be a sequence of iid (independent and identically distributed) random variables uniformly distributed on $(0,1)$. Then, for each $n = 1, 2, 3, ...$, let $\mathbf{V}_n$ be the random vector defined by $\mathbf{V}_n = (U_{nd+1}, U_{nd+2}, \dots, U_{(n+1)d})$, and let X_n be the random variable defined by $X_n = (\mathbf{1}_D f)(\mathbf{V}_n)$.

   (i) Show that the sequence $(S_n)_{n\geq 1}$ given by $S_n = \sum_{k=1}^n X_k/n$ converges almost surely towards a limit L that you will identify.

   (ii) Let now f be bounded by $M > 0$, show that for any $\lambda > 0$,

$$\mathbb{P}(|S_n - L| \geq \lambda) \leq \frac{M^2}{n\lambda^2}.$$

8. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of reals such that $\lim_{n\to+\infty} a_n = a$ and $\lim_{n\to+\infty} b_n = b$, with $a < b$. Let $(X_n)_{n\geq 1}$ be a sequence of random variables converging in distribution to a random variable X_∞ having a continuous distribution function F_∞. Prove that

$$\lim_{n\to+\infty} \mathbb{P}(X_n \in [a_n, b_n]) = \mathbb{P}(X_\infty \in [a, b]).$$