Differential Equations Comprehensive Exam
Spring 2024

Student Number: [ ]

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. If $u(x,t) \in C^2([0,1]\times[0,\infty))$ is a classical solution to the following problem

$$\begin{cases} u_{tt} - u_{xx} + u_t = -\frac{1}{4}u, \ x \in (0,1), \ t > 0, \\ u(0,t) = u(1,t) = 0, \ t > 0 \\ u(x,0) = g(x), \ u_t(x,0) = h(x), \ for \ x \in (0,1), \end{cases}$$

for any bounded C^2 functions $g(x)$ and $h(x)$ such that

$$g(0) = g(1) = h(0) = h(1) = 0.$$

Prove that

$$\lim_{t\to\infty} \|u(x,t)\|_{L^2_x([0,1])} = 0,$$

at an order like $e^{-\frac{1}{2}t}$.

2. Let B be a unit ball in $\mathbf{R}^n$ $(n \geq 2)$ centered at the origin, and u be the smooth solution of the following problem

$$\begin{cases} -\Delta u = f, \ x \in B, \\ u = g, on \ \partial B. \end{cases}$$

Prove that there exists a constant C, depending only on n, such that

$$\max_B |u| \leq C(\max_{\partial B} |g| + \max_B |f|).$$

3. For the following problem

$$\begin{cases} u_t + uu_x = 0, \ x \in \mathbf{R}, t > 0, \\ u(x,0) = \dfrac{1}{1+x^2}, \end{cases}$$

find the time and the point where the C^1 solution $u(x,t)$ blows up first.

4. Assume that $u(x,t) \in C^{2,1}((0,\pi)\times(0,\infty))$ solves

$$\begin{cases} u_t - u_{xx} = \frac{1}{4}u, \ x \in (0,\pi), \ t > 0, \\ u(0,t) = u(\pi,t) = 0, \ t > 0, \\ u(x,0) = f(x), \ for \ x \in (0,\pi), \end{cases}$$

where $f(x) \in C_0^\infty(0,\pi)$, that is $f(x)$ has compact support in $(0,\pi)$. Prove that

$$\lim_{t\to\infty} \|u(x,t)\|_{L^2_x([0,\pi])} = 0.$$

5. Consider the equation
$$\dot{y}(t) = y^2 \cos t - e^t y. \tag{1}$$
Show that for every y_0 there exists $\tau(y_0) > 0$ and a solution $Y(t, y_0)$ of (1) with $Y(0, y_0) = y_0$ defined for $t \in (-\tau(y_0), \tau(y_0))$. Can you chose $\tau(y_0)$ such that $\inf_{y_0 \in \mathbb{R}} \tau(y_0) > 0$?

6. Let $f(t)$ be defined for $t \in]1, \infty]$, continuous and such that
$$\int_1^\infty t^{n-1} |f(t)| dt < \infty$$
Show that there exists a unique solution of
$$\frac{d^n}{dt^n} y(t) + f(t) y(t) = 0$$
such that
$$\lim_{t\to\infty} y(t) = 1$$
$$\lim_{t\to\infty} y^{(i)}(t) = 0 \qquad i = 1, \ldots, n-1$$
where $y^{(i)}(t) = d^i y(t)/dt^i$.
(**Hint:** for $n = 2$ we have the integral representation
$$y(t) = y(a) + y'(a)t - \int_a^t (t-s) f(s) y(s) ds.$$
Generalize this formula for any n and then iterate it.)

7. Show that the system of equation
$$\begin{cases} \dot{x} = x - y - \left(x^2 + \frac{3}{2} y^2\right) x \\ \dot{y} = x + y - \left(x^2 + \frac{1}{2} y^2\right) y \end{cases}$$
admit at least a periodic orbit.

8. Let A be a $n \times n$ constant matrix and $g : \mathbb{R} \mapsto \mathbb{R}^n$ be continuous and periodic of period T. Consider the differential equation:
$$\dot{x}(t) = Ax(t) + g(t). \tag{2}$$
Show that if all eigenvalues of A have non 0 real part, then there exists a unique x_0 such that the solution of (2) with $x(0) = x_0$ is periodic of period T.