Topology Comprehensive Exam
Spring 2024

Student Number:

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let M and N be a pair of submanifolds of $\mathbf{R}^n$ with $\dim(M) + \dim(N) < n$. Show that M and N become disjoint after an arbitrarily small translation of N.

2. Let X be a finite CW-complex and $X^{(i)}$ be its i-skeleton (that is the union of cells of dimension less than or equal to i). Show that the fundamental group of X depends only on $X^{(2)}$.

3. Let $p : \widetilde{X} \to X$ be a covering map, and suppose that X is path-connected. Show that for any pair of points $x, y \in X$ there is a bijection between $p^{-1}(x)$ and $p^{-1}(y)$.

4. Show that the tangent space at the identity element of the orthogonal group O(n) consists of skew-symmetric matrices.

5. Let M be a submanifold of $\mathbf{R}^n$ which is diffeomorphic to S^k. If $k + 2 < n$, show that any map $f \colon S^1 \to (\mathbf{R}^n - M)$ is homotopic in $\mathbf{R}^n - M$ to a constant map. Is this true when $n = k + 2$?

6. Let Σ_g be a surface of genus g. Show that $\pi_1(\Sigma_2)$ contains $\pi_1(\Sigma_3)$ as a normal subgroup.

7. Let S and M be smooth manifolds, and $f,\ g \colon S \to M$ be homotopic maps. Suppose that ω is a closed k from on M and S is k-dimensional. Show that $\int_S f^*\omega = \int_S g^*\omega$.

8. Show that the real projective space $\mathbf{RP}^n$ is not orientable when n is even.