Analysis Comprehensive Exam
Spring 2024

Student Number: [ ]

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1  2  3  4  5  6  7  8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.

NOTE:

• All scalars in this exam are real unless explicitly stated otherwise.

• All functions in this exam are (extended) real-valued unless explicitly stated otherwise.

• The exterior Lebesgue measure of $E \subseteq \mathbb{R}^d$ is denoted by $|E|_e$, and if $E$ is measurable then its Lebesgue measure is $|E|$.

• The characteristic function of a set $A$ is denoted by $\chi_A$. 
1. Assume that $f$ and $g$ belong to $L^1[0, 1]$, and define

$$F(x) = \int_0^x f(y) \, dy \quad \text{and} \quad G(y) = \int_0^y g(x) \, dx.$$  

You may assume without proof that $h(x, y) = f(y)g(x)$ is measurable on $[0, 1]^2 = [0, 1] \times [0, 1]$. Prove that

$$\int_0^1 F(x) g(x) \, dx = F(1) G(1) - \int_0^1 f(y) G(y) \, dy.$$  

2. For $n \geq 1$, let

$$f_n(x) = \frac{ne^{-x}}{1 + n^2x^2}, \quad \text{for } x \in [0, 1].$$

Show that $\lim_{n \to \infty} f_n(x) = 0$ for $x \in (0, 1]$, and evaluate

$$\lim_{n \to \infty} \int_0^1 f_n(x) \, dx.$$  

3. The two parts of this problem are not related.

(a) Assume that $Z \subseteq \mathbb{R}$ satisfies $|Z|_e = 0$. Prove that there exists at least one point $h \in \mathbb{R}$ such that the translated set $Z + h$ contains no rational points.

(b) Let $E \subseteq \mathbb{R}^d$ be measurable, and assume functions $f_n, f$ are measurable and finite a.e. on $E$. Prove that if $f_n$ converges in measure to $f$ and $\varphi: \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then $\varphi \circ f_n$ converges in measure to $\varphi \circ f$.

4. Let $E \subseteq \mathbb{R}^n$ have finite Lebesgue measure. Let $f_k: E \to \mathbb{R}$ for $k \geq 1$. Prove that if $\{f_k\}$ converges in measure if and only if every subsequence of $\{f_k\}$ contains another subsequence of $\{f_k\}$ that converges a.e. Hint: First prove that if $\{f_k\}$ converges in measure, then a subsequence converges a.e.

5. Assume $f$ has bounded variation on $[a, b]$, and extend $f$ to the real line by setting $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$. Prove that there exists a constant $C > 0$ such that

$$\|T_t f - f\|_1 \leq C|t|, \quad \text{for all } t \in \mathbb{R},$$

where $T_t f(x) = f(x - t)$ denotes the translation of $f$ by $t$.

6. Let $E \subseteq \mathbb{R}^p$ be a set of finite Lebesgue measure. Let $f: E \to [0, \infty)$ be measurable, and finite a.e. Prove that

$$\int_E e^f = |E| + \int_0^\infty e^t \omega_f(t) \, dt,$$

where $\omega_f$ is the distribution function of $f$, given by

$$\omega_f(t) = |\{x \in E : f(x) > t\}|.$$
7. Prove that if $f, g \in L^2(\mathbb{R})$, then
\[
\lim_{n \to \infty} \int_{\mathbb{R}} f(x) g(x + n) \, dx = 0.
\]

8. Prove that there is no norm under which the vector space $P$ of polynomials with real coefficients is a Banach space.