Analysis Comprehensive Exam
Spring 2024

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

NOTE:

- All scalars in this exam are real unless explicitly stated otherwise.
- All functions in this exam are (extended) real-valued unless explicitly stated otherwise.
- The exterior Lebesgue measure of $E \subseteq \mathbf{R}^d$ is denoted by $|E|_e$, and if E is measurable then its Lebesgue measure is $|E|$.
- The characteristic function of a set A is denoted by χ_A.

1. Assume that f and g belong to $L^1[0,1]$, and define

$$F(x) = \int_0^x f(y)\,dy \qquad \text{and} \qquad G(y) = \int_0^y g(x)\,dx.$$

You may assume without proof that $h(x,y) = f(y)\,g(x)$ is measurable on $[0,1]^2 = [0,1]\times[0,1]$. Prove that

$$\int_0^1 F(x)\,g(x)\,dx = F(1)\,G(1) - \int_0^1 f(y)\,G(y)\,dy.$$

2. For $n \geq 1$, let

$$f_n(x) = \frac{ne^{-x}}{1+n^2x^2}, \quad \text{for } x \in [0,1].$$

Show that $\lim_{n\to\infty} f_n(x) = 0$ for $x \in (0,1]$, and evaluate

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx.$$

3. The two parts of this problem are not related.

(a) Assume that $Z \subseteq \mathbf{R}$ satisfies $|Z|_e = 0$. Prove that there exists at least one point $h \in \mathbf{R}$ such that the translated set $Z + h$ contains no rational points.

(b) Let $E \subseteq \mathbf{R}^d$ be measurable, and assume functions f_n, f are measurable and finite a.e. on E. Prove that if f_n converges in measure to f and $\varphi\colon \mathbf{R} \to \mathbf{R}$ is uniformly continuous, then $\varphi \circ f_n$ converges in measure to $\varphi \circ f$.

4. Let $E \subset \mathbf{R}^n$ have finite Lebesgue measure. Let $f_k : E \to \mathbf{R}$ for $k \geq 1$. Prove that $\{f_k\}$ converges in measure if and only if every subsequence of $\{f_k\}$ contains another susbsequence of $\{f_k\}$ that converges a.e. Hint: First prove that if $\{f_k\}$ converges in measure, then a subsequence converges a.e.

5. Assume f has bounded variation on $[a,b]$, and extend f to the real line by setting $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$. Prove that there exists a constant $C > 0$ such that

$$\|T_t f - f\|_1 \leq C|t|, \qquad \text{for all } t \in \mathbf{R},$$

where $T_t f(x) = f(x-t)$ denotes the translation of f by t.

6. Let $E \subset \mathbf{R}^p$ be a set of finite Lebesgue measure. Let $f : E \to [0,\infty)$ be measurable, and finite a.e. Prove that

$$\int_E e^f = |E| + \int_0^\infty e^t\,\omega_f(t)\,dt,$$

where ω_f is the distribution function of f, given by

$$\omega_f(t) = |\{x \in E : f(x) > t\}|.$$

7. Prove that if f, $g \in L^2(\mathbf{R})$, then

$$\lim_{n\to\infty} \int_{\mathbf{R}} f(x)\, g(x+n)\, dx \;=\; 0.$$

8. Prove that there is no norm under which the vector space P of polynomials with real coefficients is a Banach space.