# Algebra Comprehensive Exam
# Spring 2024

**Student Number:** ☐

*Instructions:* Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $G$ be a group of order 24, and suppose that for all $g \in G$ the order of the centralizer of $g$ is divisible by 3. Show that $G$ has a nontrivial center.

2. Suppose $G$ is a group of order $q^r p^s$ where $p$ and $q$ are primes with $q > p^s$, where $r, s \geq 1$. Show that $G$ contains a subgroup of order $q^r p$.

3. A matrix $A$ is called *idempotent* if $A^2 = A$. Show that an idempotent matrix $A$ with complex entries is diagonalizable.

4. For an ideal $J$ of a commutative ring $T$, let $\sqrt{J}$ denote the *radical* of $J$, which is the ideal $T$ consisting of $t \in T$ such that $t^m \in J$ for some integer $m \geq 1$:

$$\sqrt{J} = \{t \in T \mid t^m \in J,\ m \in \mathbf{Z},\ m \geq 1\}.$$

   Suppose $R$ and $S$ are commutative rings with unit elements, and assume there is a surjective homomorpism $\varphi : R \to S$. If $I$ is an ideal of $R$ containing $\ker(\varphi)$, prove that $\varphi(\sqrt{I}) = \sqrt{\varphi(I)}$.

5. Suppose that $p$ is a Fermat prime, i.e., $p$ has the form $2^r + 1$ for some positive integer $r$, and let $F$ denote the field with $p$ elements. Let $a, b$ be two elements of the multiplicative group $F^\times$. Show that either $a = b^n$ for some integer $n$, or $b = a^m$ for some integer $m$.

6. Let $L = \mathbf{Q}(\sqrt[3]{3}, \sqrt[5]{7})$. What is the degree of the extension $L : \mathbf{Q}$? What is the automorphism group $\mathrm{Aut}(L : \mathbf{Q})$?

7. Let $M$ be a left $R$-module where $R$ is a principal ideal domain (with unit element). For any $r \in R$, let $M(r) := \{m \in M\ :\ rm = 0\}$; that is, it is the submodule of elements of $M$ annihilated by $r$. (For the purposes of this exam you can assume this is a submodule of $M$.) If $r_1, r_2 \in R$ satisfy $(r_1, r_2) = R$ (note that $(x, y)$ denotes the ideal generated by $x$ and by $y$), prove that $M(r_1 r_2) \cong M(r_1) \oplus M(r_2)$.

8. Let $f(x) = x^4 - 2x^2 + 2 \in \mathbf{Q}[x]$. Show that $[L : \mathbf{Q}] = 4$ or $[L : \mathbf{Q}] = 8$, where $L$ is the splitting field of $f(x)$ over $\mathbf{Q}$.