Discrete Mathematics Comprehensive Exam
Spring 2024

Student Number:

Instructions: Complete **exactly 5** of the given 6 problems and **circle** their numbers below. The uncircled problems will **not** be graded.

1 2 3 4 5 6

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

All graphs in the following problems are finite and simple.

1. Let G be a nonempty graph with average degree at least 16. Show that G has a subgraph H such that H has minimum degree at least 3 and every set of 3 vertices in H is contained in an even cycle of H.

2. Let G be a graph that does not contain an induced copy of $K_{1,4}$. Show that

$$\chi(G) \leqslant \binom{\omega(G)+2}{3},$$

where $\omega(G)$ is the clique number of G.

3. Let k be an integer with $k \geqslant 2$ and let G be a connected graph such that G does not contain K_k as a minor. Show that $\chi(G) < 2^k$.

4. Fix integers $n \geqslant k \geqslant 10$. A set $X \subseteq \{0,1\}^n$ of binary strings of length n is called k-**full** if for every k distinct indices $i_1, \ldots, i_k \in [n]$, the set $\{(x_{i_1}, \ldots, x_{i_k}) : (x_1, \ldots, x_n) \in X\}$ contains all 2^k binary strings of length k. Prove that there exists a k-full set $X \subseteq \{0,1\}^n$ of size at most $\lceil k2^k \log n \rceil$. (The logarithm is base e.)

5. Fix integers $n \geqslant k \geqslant 10$. Let $a_1, \ldots, a_n$ be real numbers with $|a_i| \leqslant 1$ for all $i \in [n]$. Show that there exists a choice of signs $\epsilon_1, \ldots, \epsilon_n \in \{+1, -1\}$ such that for all $1 \leqslant i \leqslant n-k+1$,

$$|\epsilon_i a_i + \epsilon_{i+1} a_{i+1} + \cdots + \epsilon_{i+k-1} a_{i+k-1}| \leqslant 10\sqrt{k \log k}.$$

(The logarithm is base e.)

6. Let $G \sim \mathbb{G}(n, 1/2)$ be the n-vertex Erdős–Rényi random graph with edge probability $1/2$. Let $\Delta(G)$ denote the maximum degree of G. Show that

$$\mathbb{P}\big[\Delta(G) \leqslant n/2\big] \;\geqslant\; 2^{-n}.$$