Probability Comprehensive Exam
Fall 2024

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8 9

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let Ω be a sample space, let $\mathcal{F}$ be a field of subsets of Ω and let $\mathbb{P}$ be a probability measure on $\mathcal{F}$. Assume now that $A_1, A_2, \dots \in \mathcal{F}$ are nearly disjoint in the sense that $\mathbb{P}(A_i \cap A_j) = 0$, for $i \neq j$ and are such that $\cup_{n=1}^{+\infty} A_n \in \mathcal{F}$. Is it true that
$$\mathbb{P}(\cup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty} \mathbb{P}(A_n)?$$

2. For any real x, let $\{x\}$ denote the fractional part of x; that is $\{x\}$ is the unique real in $[0,1)$ such that $x - \{x\}$ is an integer. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = (0,1]$, $\mathcal{F}$ is the associated Borel σ-field and $\mathbb{P}$ is the Lebesgue measure.
   (i) Show that $\mathbb{P}(\omega \in \Omega : \{n\omega\} \leq 1/2$ for infinitely many integers $n) \geq 1/2$.
   (ii) Show that $\mathbb{P}(\omega \in \Omega : \{n\omega\} \leq 1/n \log^2 n$ for infinitely many integers $n) = 0$.

3. Let $\{X_n\}$ be a sequence of random variables with values in $[0,1]$ and such that, for some $a > -1$,
$$\mathbb{E}X_n^k \to \frac{a+1}{a+1+k} \text{ as } n \to \infty, \; k = 0, 1, 2, \dots.$$
Prove that $\{X_n\}$ converges in distribution to a random variable X and find the distribution of X.

4. Let $X, X_1, \dots, X_n, \dots$ be i.i.d. random variables and let $S_n := X_1 + \cdots + X_n$. Suppose, for some $p > 1$, $\mathbb{E}|X|^{1/p} < \infty$. Prove that
$$\frac{S_n}{n^p} \to 0 \text{ as } n \to \infty \text{ a.s.}$$

5. Let X, Y be independent random variables with $\mathbb{E}X^2 < \infty, \mathbb{E}Y^2 < \infty$. Show that, if $X+Y$ and $X-Y$ are independent, then X and Y are both normal random variables and $\mathrm{Var}(X) = \mathrm{Var}(Y)$.

6. Let X be a real valued random variable with density p and let $f : \mathbb{R} \mapsto \mathbb{R}$ be a Borel measurable function such that $\mathbb{E}|f(X)| < \infty$. Find $\mathbb{E}(f(X)|X^2)$.

7. Let $(X_n)_{n\geq 1}$ be a sequence of iid centered random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such that $0 < \mathbb{E}X_1^2 < +\infty$, and let $(a_n)_{n\geq 1}$ be a sequence of reals. Show that the series $\sum_{n=1}^{+\infty} a_n X_n$ converges with probability one if and only if $\sum_{n=1}^{+\infty} a_n^2 < +\infty$.

8. Let $(X_n)_{n\geq 1}$ be a sequence of iid standard normal random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(a_n)_{n\geq 0}$ be a sequence of reals such that $a_j a_{j+1} = 0$, for all $j \geq 0$, and such that $\sum_{j=0}^{\infty} a_j^2 < +\infty$. Let

$$Y_n = \sum_{k=1}^{n} a_{n-k} X_k, \quad n = 1, 2, \ldots$$

(i) Does the sequence $(Y_n)_{n\geq 1}$ converge in law? If yes, can you identify the limiting distribution?

(ii) Are the random variables Y_n and Y_{n+1}, $n \geq 1$ dependent?