# Topology Comprehensive Exam
# Fall 2024

**Student Number:** ☐

*Instructions:* Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Show that if a smooth manifold can be covered by two coordinate charts whose intersection is connected then it is orientable.

2. Let GL(n, $\mathbf{R}$) be the space of $n \times n$ real valued matrices with nonzero determinant, endowed with its standard smooth manifold structure as a subset of $\mathbf{R}^{n^2}$. Show that the determinant map $f\colon \mathrm{GL}(n, \mathbf{R}) \to \mathbf{R}$ is a submersion.

3. Let $M$ be a compact $n$-manifold without boundary smoothly embedded in $\mathbf{R}^{n+1}$. Suppose that the origin $o$ of $\mathbf{R}^n$ does not lie on $M$. Show that there exists a line passing through $o$ which intersects $M$ at most finitely many times.

4. Let $M$ be $\mathbf{S}^1 \times \mathbf{S}^{n-1}$ minus one point, where $n \geq 2$. Use intersection theory to show that there is no smooth embedding of $M$ into $\mathbf{R}^n$.

5. Show that the antipodal map $f\colon \mathbf{S}^n \to \mathbf{S}^n$, $f(x) := -x$, is homotopic to the identity if and only if $n$ is odd.

6. Let $\Sigma_g$ be the closed orientable surface of genus $g$, that is the 2-sphere with $g$ handles attached. For every $g \geq 3$, show that $\pi_1(\Sigma_g)$ is isomorphic to a normal subgroup of $\pi_1(\Sigma_2)$.

7. The Klein bottle $K$ may be described as a square with opposite sides identified. The sides of the square project to the wedge sum of two circles inside $K$. Show that $K$ retracts to one of the circle summands, but does not retract to the other circle.

8. Let $X$ be the complement of the $z$-axis in $\mathbf{R}^3$. Show that every continuous map from $\mathbf{RP}^2$ to $X$ is null-homotopic.