# Analysis Comprehensive Exam
# Fall 2024

**Student Number:** ☐

*Instructions:* Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. (a) Prove that a proper subspace $Y$ of a normed space $X$ has empty interior.

   (b) Prove that a Banach space cannot be written as a countable union of proper closed subspaces.

   (Recall that a subspace $Y$ of a vector space $X$ is called proper if $Y \neq \varnothing$ and $Y \neq X$.)

2. Suppose $\nu$ is the Borel measure on $(0, \infty)$ given by

   $$\nu(E) = \int_E \frac{dt}{t}$$

   for a Borel set $E \subset \mathbb{R}$.

   Suppose $K$ is a continuous function on $(0, \infty) \times (0, \infty)$ and there exists $\varepsilon > 0$ such that

   $$|K(s,t)| \leq \min\Big(\frac{s}{t}, \frac{t}{s}\Big)^{\varepsilon} \text{ for every } s, t \in (0, \infty).$$

   Prove that the operator

   $$T(f)(t) = \int_0^\infty K(s,t) f(s) d\nu(s)$$

   is bounded from $L^p(\nu)$ to $L^p(\nu)$ for every $1 < p < \infty$.

3. For any Borel set $A \subset \mathbf{R}^2$ with $|A| < \infty$ (where $|A|$ denotes Lebesgue measure), prove that

   $$\int_A \frac{1}{|y|} dy \leq 2\sqrt{\pi |A|}.$$

4. Suppose that $(X, \mathcal{A}, \mu)$ is a $\sigma$-finite measure space.

   (a) Prove that if $E \in \mathcal{A}$ satisfies $\mu(E) > 0$, then there exists $F \subset E$, $F \in \mathcal{A}$ with $0 < \mu(F) < \infty$.

   (b) Prove that if $f$ and $g$ are real valued measurable functions that satisfy $\int_E f d\mu = \int_E g d\mu$ for every $E \in \mathcal{A}$, then $f = g$ $\mu$-almost everywhere.

   (c). Give an example to show that the conclusion of part (b) can fail without the assumption of $\sigma$-finiteness.

5. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be Lebesgue integrable.
   (a) Prove that there is a sequence of polynomials $\{p_n\}$ such that for a.e. $x \in \mathbb{R}$,

   $$\lim_{n \to \infty} p_n(x) = f(x).$$

(Hint: You may assume that for each finite $b > a$ and $\varepsilon > 0$, there exists a polynomial $p$ such that

$$\int_a^b |f(x) - p(x)|\, dx < \varepsilon.)$$

(b) Deduce that given $\delta > 0$, and a compact interval $[a, b]$, you can find a set $F \subset [a, b]$ of measure $< \delta$ such that $\{p_n\}$ converges uniformly to $f$ on $[a, b] \setminus F$.

6. Let $0 < \beta < 1$.
For measurable sets $F \subset \mathbb{R}$, let $|F|$ denote the Lebesgue measure of $F$.
(a) Construct a measurable set $E$ in $[-1, 1]$ such that

$$\limsup_{\delta \to 0+} \frac{|E \cap [-\delta, \delta]|}{2\delta} = \beta$$

but

$$\liminf_{\delta \to 0+} \frac{|E \cap [-\delta, \delta]|}{2\delta} = 0.$$

(b) What does Lebesgue's differentiation theorem say about

$$\lim_{\delta \to 0+} \frac{|E \cap [x - \delta, x + \delta]|}{2\delta}$$

for a.e. $x \in E$? Just state your answer for (b), do not prove it.

7. Let $\mu$ and $\nu$ be measures on $[0, \infty)$ with finite total mass, so that $\mu([0, \infty)) < \infty$ and $\nu([0, \infty)) < \infty$. Let $r \in (0, 1)$, $s > 0$ and $\omega$ be the measure defined by

$$\omega = r\mu + s\nu.$$

(a) Show that $\mu$ is absolutely continuous with respect to $\omega$.

(b) Let $g$ denote the Radon-Nikodym derivative of $\mu$ with respect to $\omega$, so that

$$\int f \, d\mu = \int fg \, d\omega$$

for every integrable function $f$. Show that

$$0 \le g \le \frac{1}{r} \text{ a.e. } (\mu)$$

8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be integrable in $\mathbb{R}^n$. Let $K : \mathbb{R}^n \to [0, \infty)$ be nonnegative, measurable, and bounded in $\mathbb{R}^n$, with $\int_{\mathbb{R}^n} K = 1$ and $K(\mathbf{t}) = 0$ for $|\mathbf{t}| \geq 1$. For $h > 0$, and $\mathbf{x} \in \mathbb{R}^n$, define

$$\Phi_h[f](\mathbf{x}) = h^{-n} \int_{\mathbb{R}^n} f(\mathbf{x} + \mathbf{t}) K\left(\frac{\mathbf{t}}{h}\right) d\mathbf{t}.$$

and

$$\Omega(f; h) = \sup_{|\mathbf{t}| \leq h} \int_{\mathbb{R}^n} |f(\mathbf{x} + \mathbf{t}) - f(\mathbf{x})| \, d\mathbf{x}.$$

(a) Prove that

$$\int_{\mathbb{R}^n} |\Phi_h[f](\mathbf{x}) - f(\mathbf{x})| \, d\mathbf{x} \leq \Omega(f; h).$$

(b) Prove that

$$\lim_{h \to 0+} \Omega(f; h) = 0.$$