Algebra Comprehensive Exam
Fall 2024

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let G be a finite group with $|G| \geq n!$ for some positive integer n. Show that if G has a subgroup of index at most n, then G is not simple.

2. Suppose G is a finite p-group and $N \neq 1$ is a normal subgroup of G. Prove that $N \cap Z(G) \neq 1$, where $Z(G)$ denotes the center of G.

3. We say that a ring R is **Artinian** if every descending chain $I_1 \supseteq I_2 \supseteq \cdots$ of ideals of R stabilizes, i.e, there exists N such that $I_n = I_{n+1}$ for all $n \geq N$. Prove that a commutative Artinian integral domain must be a field.

4. Let A be a $\mathbf{Z}$-module with generators a, b, c and relations $3a+4b+3c = 0$, $4a+2b+4c = 0$, $5a + 2b + 3c = 0$. Write A as a direct sum of cyclic modules. Explain your steps clearly.

5. Let A be a 3×3 matrix with rational entries which satisfies $A^5 = I$ where I is the 3×3 identity matrix. Show that $A = I$.

6. Let $\alpha = \sqrt{2 + \sqrt{2}}$. Show that $\mathbf{Q}(\alpha)/\mathbf{Q}$ is a Galois extension, and determine the size of the Galois group. (You may find it helpful to note that $\sqrt{2+\sqrt{2}} \cdot \sqrt{2 - \sqrt{2}} = \sqrt{2}$).

7. Prove or disprove: $f(x) = x^4 + 3x + 3$ is irreducible over the field $K = \mathbf{Q}(\sqrt[3]{2})$.

8. Suppose $f \in \mathbf{Q}[x]$ is a monic irreducible polynomial with both a real root and a complex root. Prove that the Galois group of f is non-abelian. (Hint: Recall that the Galois group acts transitively on the roots of f).