# Discrete Mathematics Comprehensive Exam
# Fall 2024

**Student Number:** ☐

*Instructions:* Complete **exactly 5** of the given 6 problems and **circle** their numbers below. The uncircled problems will **not** be graded.

1 2 3 4 5 6

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

All graphs in the following problems are finite and simple.

1. Let $m, n$ be positive integers such that $m \geqslant 3n-5$ and $n \geqslant 3$. Let $G$ be a connected graph with $n$ vertices and $m$ edges. Show that for any basis of the cycle space of $G$, some edge of $G$ must appear in at least three members of that basis.

2. Let $T$ be a tree with at least one edge. Show that if a graph $G$ does not contain a copy of $T$ as a subgraph, then $\chi(G) \leqslant |V(T)| - 1$. (Hint: Bound the minimum degree of $G$.)

3. Let $n$, $r$ be integers with $n \geqslant r+1 \geqslant 4$ and let $G$ be a graph on $n$ vertices with $|E(G)| > t_{r-1}(n)$, where $t_{r-1}(n) = |E(T_{r-1}(n))|$ and $T_{r-1}(n)$ is the balanced complete $(r-1)$-partite graph on $n$ vertices. Show that $G$ contains two $r$-cliques sharing $r-1$ vertices.

4. Fix a graph $F$ and suppose that there exists a graph $H$ with $n$ vertices, $m$ edges, and no subgraph isomorphic to $F$. Show that if $q \geqslant (n^2 \log n)/m$, then the edges of the complete graph $K_n$ can be colored using $q$ colors so that no copy of $F$ is monochromatic. (The logarithm is base $e$.)

5. Suppose that $p = p(n) \in (0,1)$ is a function such that $p \gg n^{-1}$ and $1-p \gg n^{-2}$. Show that the Erdős–Rényi random graph $G \sim \mathbb{G}(n,p)$ has an **induced** 4-cycle with high probability. (Hint: Consider the cases $p \leqslant 1/2$ and $p > 1/2$ separately.)

6. Show that there is an absolute constant $c > 0$ with the following property. Let $G_1, \ldots, G_t$ be 5-regular graphs with the same vertex set $V$, where $|V| = n \geqslant 6$ and $t \leqslant 2^{cn}$. Then there is a partition $(A, B)$ of $V$ such that each $G_i$ has at least $n$ edges joining $A$ to $B$.