Probability Comprehensive Exam
Spring 2023

Student Number:

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $X_1, X_2, \ldots$ be i.i.d. random variables that are uniformly distributed on the interval $[-1, 1]$. Does the following limit exist almost surely:
$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{X_i}?$$
Prove or disprove.

2. a) Let X, Y be identically distributed random variables taking two values; that is, there exist $a, b \in \mathbb{R}$ with $a < b$ such that $\mathbb{P}(X \in \{a, b\}) = 1$. Suppose that $\mathbb{E}XY = \mathbb{E}X\mathbb{E}Y$. Show that X and Y are independent.
   b) Let X, Y be identically distributed random variables taking three values; that is, there exist $a, b, c \in \mathbb{R}$ such that $\mathbb{P}(X \in \{a, b, c\}) = 1$. Suppose that $\mathbb{E}XY = \mathbb{E}X\mathbb{E}Y$. Must X and Y be independent?

3. Prove or disprove: there exist i.i.d. random variables X, Y such that the random variable $X - Y$ is uniformly distributed on the interval $[-1, 1]$.

4. Let (X_n) be a sequence of random variables with corresponding distribution functions (F_n). Let X be another random variable with distribution function F. Show that the following are equivalent.
   1. (X_n) converges to X in distribution.
   2. There exists a dense subset S of $\mathbb{R}$ such that $F_n(x) \to F(x)$ as $n \to \infty$ for every $x \in S$.

5. Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables. Prove or disprove the following equivalence $\mathbb{P}(\sup_n X_n < +\infty) = 1$ if and only if there exists a positive real K such that $\sum_{n=1}^{\infty} \mathbb{P}(X_n > K) < +\infty$.

6. Let X and Y be two bounded random variables such that for all $k = 0, 1, 2, 3, \ldots$ and all $\ell = 0, 1, 2, 3, \ldots$,
$$\mathbb{E}(X^k Y^\ell) = \mathbb{E}(X^k)\mathbb{E}(Y^\ell).$$
Are X and Y independent?

7. Let $\mathbb{P}_n$ and $\mathbb{P}_\infty$ be probability measures which are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and with respective densities f_n and f_∞.
   (i) Show that if $\lim_{n\to\infty} f_n(x) = f_\infty(x)$ almost everywhere (Leb.), then as $n \to +\infty$,
$$\mathbb{P}_n \Longrightarrow \mathbb{P}_\infty;$$

that is, $\mathbb{P}_n$ converges to $\mathbb{P}_\infty$ weakly.

(ii) Is the converse implication to the statement (i) above true?

8. Let $X_1, X_2, \ldots,$ be centered i.i.d. random variables with finite variance equal to 1. Let

$$Z_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k,\ \ n = 1, 2, \ldots$$

Can the sequence $(Z_n)_{n\geq 1}$ converge almost surely to a random variable Z?