Probability Comprehensive Exam
Spring 2023

Student Number: [ ]

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. Let \( X_1, X_2, \ldots \) be i.i.d. random variables that are uniformly distributed on the interval \([-1, 1]\). Does the following limit exist almost surely:
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{X_i}?
\]
Prove or disprove.

2. a) Let \( X, Y \) be identically distributed random variables taking two values; that is, there exist \( a, b \in \mathbb{R} \) with \( a < b \) such that \( \mathbb{P}(X \in \{a, b\}) = 1 \). Suppose that \( \mathbb{E}XY = \mathbb{E}X \mathbb{E}Y \). Show that \( X \) and \( Y \) are independent.

b) Let \( X, Y \) be identically distributed random variables taking three values; that is, there exist \( a, b, c \in \mathbb{R} \) such that \( \mathbb{P}(X \in \{a, b, c\}) = 1 \). Suppose that \( \mathbb{E}XY = \mathbb{E}X \mathbb{E}Y \). Must \( X \) and \( Y \) be independent?

3. Prove or disprove: there exist i.i.d. random variables \( X, Y \) such that the random variable \( X - Y \) is uniformly distributed on the interval \([-1, 1]\).

4. Let \((X_n)\) be a sequence of random variables with corresponding distribution functions \((F_n)\). Let \( X \) be another random variable with distribution function \( F \). Show that the following are equivalent.
   1. \((X_n)\) converges to \( X \) in distribution.
   2. There exists a dense subset \( S \) of \( \mathbb{R} \) such that \( F_n(x) \to F(x) \) as \( n \to \infty \) for every \( x \in S \).

5. Let \((X_n)_{n \geq 1}\) be a sequence of independent random variables. Prove or disprove the following equivalence \( \mathbb{P}(\sup_n X_n < +\infty) = 1 \) if and only if there exists a positive real \( K \) such that \( \sum_{n=1}^{\infty} \mathbb{P}(X_n > K) < +\infty \).

6. Let \( X \) and \( Y \) be two bounded random variables such that for all \( k = 0, 1, 2, 3, \ldots \) and all \( \ell = 0, 1, 2, 3, \ldots \),
\[
\mathbb{E}(X^k Y^\ell) = \mathbb{E}(X^k) \mathbb{E}(Y^\ell).
\]
Are \( X \) and \( Y \) independent?

7. Let \( \mathbb{P}_n \) and \( \mathbb{P}_\infty \) be probability measures which are absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R} \) and with respective densities \( f_n \) and \( f_\infty \).
   i) Show that if \( \lim_{n \to \infty} f_n(x) = f_\infty(x) \) almost everywhere (Leb.), then as \( n \to +\infty \),
\[
\mathbb{P}_n \Longrightarrow \mathbb{P}_\infty;
\]
that is, $\mathbb{P}_n$ converges to $\mathbb{P}_\infty$ weakly.

(ii) Is the converse implication to the statement (i) above true?

8. Let $X_1, X_2, \ldots$, be centered i.i.d. random variables with finite variance equal to 1. Let

$$Z_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k, \quad n = 1, 2, \ldots$$

Can the sequence $(Z_n)_{n \geq 1}$ converge almost surely to a random variable $Z$?