Discrete Mathematics Comprehensive Exam
Spring 2023

Student Number: ☐

Instructions: Complete **exactly 5** of the given 6 problems and **circle** their numbers below. The uncircled problems will **not** be graded.

1   2   3   4   5   6

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

All graphs in this exam are finite and simple.

1. Show that for integers $s \geqslant 2$ and $t \geqslant 2$, there is a graph G with chromatic number

$$\chi(G) \geqslant \frac{R(s, t+1) - 1}{s - 1}$$

that contains neither $K_{1,s}$ nor K_{t+1} as induced subgraphs. (Here $R(r_1, r_2)$ is the minimum integer n such that every graph of order at least n has an independent set of size r_1 or a clique of size r_2.)

2. Let G be a 3-edge-connected graph and let H be a bipartite subgraph of G with the most edges. Prove that H is 2-edge-connected.

3. Suppose G is a graph that does not contain any K_4-subdivision. Show that $\chi(G) \leqslant 3$. (Hint: You could start by proving that every 3-connected graph contains a K_4-subdivision.)

4. For a natural number n, let C_n denote the expected number of cycles in a uniformly random permutation of the set $[n]$. Show that $C_n \sim \log n$, i.e., the ratio $C_n / \log n$ tends to 1 as $n \to \infty$. (The logarithm is base e.)

5. Prove that the unit sphere in $\mathbb{R}^n$ contains a set of at least $e^{n/100}$ points all whose pairwise distances are at least 1. ('Distance' here means the usual Euclidean distance in $\mathbb{R}^n$.)

6. Recall that a family $\mathcal{F}$ of sets is *decreasing* if for every $A \in \mathcal{F}$, all subsets of A also belong to $\mathcal{F}$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be decreasing families of subsets of $[n]$ such that $|\mathcal{F}_i| = 2^{n-1}$ for all i. Show that $\left|\bigcap_{i=1}^k \mathcal{F}_i\right| \geqslant 2^{n-k}$ and $\left|\bigcup_{i=1}^k \mathcal{F}_i\right| \leqslant 2^n - 2^{n-k}$.