Numerical Analysis Comprehensive Exam
Spring 2023

Student Number:

Instructions: Complete 5 of the 7 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Consider the matrix
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

   (i) Give a Householder transformation $Q \in \mathbb{R}^{3\times 3}$ such that QA has the upper Hessenberg form
$$QA = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \times & \times & \times \\ 0 & \times & \times \end{bmatrix} \in \mathbb{R}^{3\times 3}.$$

   (ii) What are the eigenvalues and eigenvectors of Q?

2. Let $f(x)$ be a sufficiently smooth function defined on $[a, b]$. Show that the Simpson's rule is 5th order accurate, i.e.
$$\int_a^b f(x)dx = \frac{h}{6}\{f(a) + 4f(\frac{a+b}{2}) + f(b)\} + O(h^5),$$
where $h = b - a$.

3. Consider solving an ODE: $y_t = f(t, y)$. Write down the improved Euler method and show that it's second order accurate.

4. Write down an explicit upwind scheme for solving the initial-value problem $u_t + a(x)u_x = 0$ with the initial $u(x, 0) = f(x)$ given, where $a(x)$ and $f(x)$ are differentiable functions defined on R, and $|a(x)| \leq M$ for any $x \in R$, M is a positive constant. Find an upper bound for $\Delta t/\Delta x$ so that it's stable, and prove your result.

5. Consider the 2D heat equation $u_t = u_{xx} + u_{yy}$. The Crank-Nicolson scheme for solving it can be written as
$$\begin{aligned} \frac{U^{n+1}_{j,k} - U^n_{j,k}}{\Delta t} &= \frac{1}{2}\left\{\frac{U^n_{j+1,k} - 2U^n_{j,k} + U^n_{j-1,k}}{\Delta x^2} + \frac{U^{n+1}_{j+1,k} - 2U^{n+1}_{j,k} + U^{n+1}_{j-1,k}}{\Delta x^2}\right\} + \\ &\quad \frac{1}{2}\left\{\frac{U^n_{j,k+1} - 2U^n_{j,k} + U^n_{j,k-1}}{\Delta y^2} + \frac{U^{n+1}_{j,k+1} - 2U^{n+1}_{j,k} + U^{n+1}_{j,k-1}}{\Delta y^2}\right\}. \end{aligned}$$

   Analyze the truncation error and stability of the Crank-Nicolson scheme.

6. The hyperbolic function sinh has the following Taylor series:

$$\sinh(x) = \sum_{k=0} \frac{x^{2k+1}}{(2k+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots, \quad \text{for all } x.$$

   Consider the algorithm of computing sinh(1) by summing

$$\sum_{k=0} \frac{1}{(2k+1)!}$$

   from left (small k) to right (large k) using floating point operations $\otimes$ and $\oplus$, stopping when a summand is reached of magnitude $< \epsilon_{\text{machine}}$. State whether the algorithm is backward stable, stable but not backward stable, or unstable and prove your statement. You might need the Stirling's approximation $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

7. Assume that the unit vector $\vec{x}$ ($\|\vec{x}\| = 1$) is sufficiently close to an eigenvector $\vec{q}$ of a symmetric matrix $A \in \mathbb{R}^{n\times n}$ with the corresponding eigenvalue λ. Assume that the eigenvalue λ has geometric multiplicity 1.

   (i) Describe the algorithm of the Rayleigh quotient iteration for the computation of the eigenvalue λ and the eigenvector $\vec{q}$.

   (ii) When the Rayleigh quotient iteration converges, what is the rate of convergence? Is it linear, quadratic or cubic? Explain your statement.