Topology Comprehensive Exam
Spring 2023

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

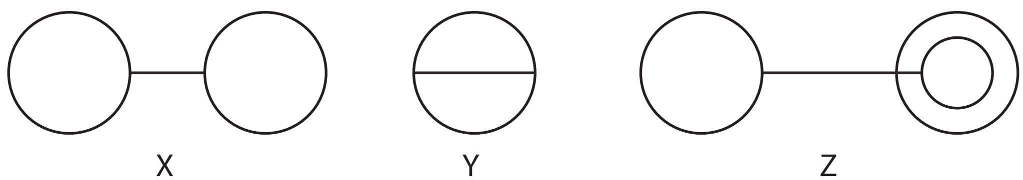
1. Show that if M and N are smooth manifolds, then so is $M \times N$.

2. Let T^3 be the 3-torus, that is the product of three copies of S^1, and let $\mathbf{RP}^3$ be the real projective 3-space. Compute the fundamental groups of T^3 and $\mathbf{RP}^3$. Show that neither space is a covering space of the other.

3. Let M be a smooth closed submanifold of Euclidean space $\mathbf{R}^n$, and $u \in \mathbf{S}^{n-1}$ be a unit vector. Show that almost every hyperplane $H \subset \mathbf{R}^n$ which is orthogonal to u is transversal to M.

4. Suppose that M is a compact smooth manifold and α is a closed 1-form on M. Show that if α is never zero, then the first De Rham cohomology group $H^1_{DR}(M)$ is non-trivial.

5. The three spaces, X, Y, and Z in the figure below are subsets of $\mathbf{R}^2$. Determine, with proof, which are homotopy equivalent and which are not. In addition, determine which are homeomorphic and which are not. State any theorem you are using in your proof.



6. Let $p_n \colon S^1 \to S^1$ be the map given by $p_n(\theta) = n\theta$, and X be the space obtained from the disjoint union of S^1 and $S^1 \times [0,1]$ by gluing $S^1 \times \{0\}$ to S^1 by p_2 and gluing $S^1 \times \{1\}$ to S^1 by p_3. Find a presentation for the fundamental group $\pi_1(X)$.

7. Let M be a compact orientable smooth manifold with boundary ∂M. Use Stokes theorem to show that there exists no smooth retraction $r \colon M \to \partial M$.

8. Show that the special linear group $SL(n)$ is a smooth manifold and compute its dimension.