Analysis Comprehensive Exam
Spring 2023

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

NOTE:

- All scalars in this exam are real unless explicitly stated otherwise.
- All functions in this exam are (extended) real-valued unless explicitly stated otherwise.
- The exterior Lebesgue measure of $E \subseteq \mathbf{R}^d$ is denoted by $|E|_e$, and if E is measurable then its Lebesgue measure is $|E|$.
- The characteristic function of a set A is denoted by χ_A.

1. (a) Suppose that f is a *monotone increasing* function that is absolutely continuous on $[0,1]$. Prove that if U is any open subset of $[0,1]$, then

$$|f(U)| \;=\; \int_U f'(x)\,dx.$$

(b) Give an example that shows that part (a) need not hold if f is continuous but not absolutely continuous.

2. The two parts of this problem are not related.

(a) Assume that functions $f_n \in L^1(\mathbf{R})$ are such that $f_n \to f$ a.e., and there exists some $M \geq 0$ such that

$$\int_{-\infty}^{\infty} \max\{|f_1|,\dots,|f_n|\} \;<\; M, \qquad \text{for every } n \in N.$$

Prove that $f_n \to f$ in L^1-norm, i.e., $\|f - f_n\|_1 \to 0$ as $n \to \infty$.

(b) Prove that if $f \in L^p(\mathbf{R})$, where $1 \leq p < \infty$, then $\lim_{k\to\infty} k^p \,\big|\{x \in \mathbf{R} : |f(x)| > k\}\big| \;=\; 0.$

3. Prove that

$$F(t) \;=\; \int_{-\infty}^{\infty} \frac{\sin xt}{1+x^4}\,dx$$

is differentiable on $(0,\infty)$.

4. Prove that there is a unique function $f \in C[0,1]$ that satisfies

$$f(x) \;=\; \int_0^x t\,f(t)\,dt, \qquad \text{for } x \in [0,1].$$

5. Let E be a Lebesgue measurable subset of $\mathbf{R}^d$. Assume that:

(a) $f_n,\ g_n,\ f,\ g \in L^1(E)$,

(b) $f_n \to f$ pointwise a.e.,

(c) $g_n \to g$ pointwise a.e.,

(d) $|f_n| \leq g_n$ a.e., and

(e) $\int_E g_n \to \int_E g$.

Prove that $\int_E f_n \to \int_E f$ and $\|f - f_n\|_1 \to 0$.

6. Let A and B be measurable subsets of $\mathbf{R}$, and let $A + B = \{a + b : a \in A, b \in B\}$.

(a) Fix $\alpha,\ \beta \in \mathbf{R}$ and $r > 0$. Prove that if $A \subseteq [\alpha, \alpha+r]$ and $B \subseteq [\beta, \beta+r]$ have measures $|A|,\ |B| > r/2$, then $\alpha+\beta+r \in A+B$.

(b) Prove that if $A + B \subseteq \mathbf{R}\backslash\mathbf{Q}$, then either $|A| = 0$ or $|B| = 0$.

7. Assume that k is a measurable function on $\mathbf{R}^2$ and there are strictly positive measurable functions u, v on $\mathbf{R}$ such that

$$\int_{-\infty}^{\infty} |k(x,y)|\, v(y)\, dy \;\leq\; C_1\, u(x), \qquad \text{for a.e. } x,$$

$$\int_{-\infty}^{\infty} |k(x,y)|\, u(x)\, dx \;\leq\; C_2\, v(y), \qquad \text{for a.e. } y.$$

Prove that operator

$$L_k f(x) \;=\; \int_{-\infty}^{\infty} k(x,y)\, f(y)\, dy, \qquad \text{for } f \in L^2(\mathbf{R}),$$

defines a bounded mapping of $L^2(\mathbf{R})$ into itself. Give a bound on the operator norm of L_k in terms of C_1 and C_2.

Note: You can assume without proof that $L_k f$ is measurable.

8. Let μ be a finite signed Borel measure on $\mathbf{R}$ that is absolutely continuous with respect to Lebesgue measure. Assume that $A \subseteq \mathbf{R}$ is a Borel set, and for $t \in \mathbf{R}$ let $A + t = \{x + t : x \in A\}$. Prove that $f(t) = \mu(A + t)$ is continuous on $\mathbf{R}$.