Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.

NOTE:

• All scalars in this exam are real unless explicitly stated otherwise.

• All functions in this exam are (extended) real-valued unless explicitly stated otherwise.

• The exterior Lebesgue measure of $E \subseteq \mathbb{R}^d$ is denoted by $|E|_e$, and if $E$ is measurable then its Lebesgue measure is $|E|$.

• The characteristic function of a set $A$ is denoted by $\chi_A$. 
1. (a) Suppose that \( f \) is a monotone increasing function that is absolutely continuous on \([0, 1]\). Prove that if \( U \) is any open subset of \([0, 1]\), then
\[
|f(U)| = \int_U f'(x) \, dx.
\]
(b) Give an example that shows that part (a) need not hold if \( f \) is continuous but not absolutely continuous.

2. The two parts of this problem are not related.
   (a) Assume that functions \( f_n \in L^1(\mathbb{R}) \) are such that \( f_n \to f \) a.e., and there exists some \( M \geq 0 \) such that
\[
\int_{-\infty}^{\infty} \max\{|f_1|, \ldots, |f_n|\} < M, \quad \text{for every } n \in \mathbb{N}.
\]
Prove that \( f_n \to f \) in \( L^1 \)-norm, i.e., \( \|f - f_n\|_1 \to 0 \) as \( n \to \infty \).
(b) Prove that if \( f \in L^p(\mathbb{R}) \), where \( 1 \leq p < \infty \), then \( \lim_{k \to \infty} k^p \|\{x \in \mathbb{R} : |f(x)| > k\} = 0 \).

3. Prove that
\[
F(t) = \int_{-\infty}^{\infty} \frac{\sin xt}{1 + x^4} \, dx
\]
is differentiable on \((0, \infty)\).

4. Prove that there is a unique function \( f \in C[0, 1] \) that satisfies
\[
f(x) = \int_0^x tf(t) \, dt, \quad \text{for } x \in [0, 1].
\]

5. Let \( E \) be a Lebesgue measurable subset of \( \mathbb{R}^d \). Assume that:
   (a) \( f_n, g_n, f, g \in L^1(E) \),
   (b) \( f_n \to f \) pointwise a.e.,
   (c) \( g_n \to g \) pointwise a.e.,
   (d) \( |f_n| \leq g_n \) a.e., and
   (e) \( \int_E g_n \to \int_E g \).

Prove that \( \int_E f_n \to \int_E f \) and \( \|f - f_n\|_1 \to 0 \).

6. Let \( A \) and \( B \) be measurable subsets of \( \mathbb{R} \), and let \( A + B = \{a + b : a \in A, b \in B\} \).
   (a) Fix \( \alpha, \beta \in \mathbb{R} \) and \( r > 0 \). Prove that if \( A \subseteq [\alpha, \alpha+r] \) and \( B \subseteq [\beta, \beta+r] \) have measures \( |A|, |B| > r/2 \), then \( \alpha + \beta + r \in A + B \).
   (b) Prove that if \( A + B \subseteq \mathbb{R} \setminus \mathbb{Q} \), then either \( |A| = 0 \) or \( |B| = 0 \).
7. Assume that $k$ is a measurable function on $\mathbb{R}^2$ and there are strictly positive measurable functions $u, v$ on $\mathbb{R}$ such that
\[
\int_{-\infty}^{\infty} |k(x, y)| v(y) \, dy \leq C_1 \, u(x), \quad \text{for a.e. } x,
\]
\[
\int_{-\infty}^{\infty} |k(x, y)| u(x) \, dx \leq C_2 \, v(y), \quad \text{for a.e. } y.
\]
Prove that operator
\[
L_k f(x) = \int_{-\infty}^{\infty} k(x, y) \, f(y) \, dy, \quad \text{for } f \in L^2(\mathbb{R}),
\]
defines a bounded mapping of $L^2(\mathbb{R})$ into itself. Give a bound on the operator norm of $L_k$ in terms of $C_1$ and $C_2$.

Note: You can assume without proof that $L_k f$ is measurable.

8. Let $\mu$ be a finite signed Borel measure on $\mathbb{R}$ that is absolutely continuous with respect to Lebesgue measure. Assume that $A \subseteq \mathbb{R}$ is a Borel set, and for $t \in \mathbb{R}$ let $A + t = \{x + t : x \in A\}$. Prove that $f(t) = \mu(A + t)$ is continuous on $\mathbb{R}$. 

