Algebra Comprehensive Exam
Spring 2023

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let A_n be the alternating group consisting of even permutations of $\{1, 2, \ldots, n\}$.
   (a) Show that all 3-cycles are conjugate in A_n for $n \geq 5$.
   (b) Are all 3 cycles conjugate in A_4?

2. Let G be a nonabelian group of order p^3 where p is a prime number.
   (a) Show that the center Z of G is a cyclic group of order p.
   (b) Show that the commutator subgroup of G is equal to Z.
   (The commutator subgroup $[G, G]$ is the subgroup generated by elements of the form $aba^{-1}b^{-1}$ where $a, b \in G$.)

3. Show that there exists a real $n \times n$ matrix A with $A^2 = -I$ if and only if n is even.

4. Let K be the splitting field of $x^4 - 3$ over $\mathbf{Q}$. What is the degree of K over $\mathbf{Q}$? How many subfields are there in K (including $\mathbf{Q}$ and K itself)?

5. Let p be a prime number, let $q = p^n$ with $n \geq 1$ a positive integer, and let $\mathbf{F}_q$ denote the finite field with q elements. Prove that every p-Sylow subgroup of the group $\mathrm{GL}_2(\mathbf{F}_q)$ of 2×2 invertible matrices over $\mathbf{F}_q$ is isomorphic to $(\mathbf{Z}/p\mathbf{Z})^n$.

6. Suppose L/K is an algebraic field extension, and that R is a subring of L containing K. Prove that R is a field.

7. Find all commutative rings R with 1 such that R has a unique maximal ideal and such that the only units of R are 1 and -1.

8. Prove that the group $\mathrm{SL}(2, \mathbf{Z})$ of 2×2 integer matrices with determinant 1 does not contain an element of order 5.