Probability Comprehensive Exam
Fall 2023

Student Number:

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let X and Y be two independent and identically distributed random variables, with finite second moment and variance equal to 1, and assume further that $(X+Y)/\sqrt{2}$ has the same law as X.

   (a) Show that X is centered.

   (b) Let X_1, X_2, Y_1, Y_2 be independent random variables each having the same law as X. Show that $(X_1 + Y_1 + X_2 + Y_2)/2$ has also the same law as X.

   (c) By induction, generalize (b) and conclude that X is in fact a standard normal random variable.

2. Let $(X_n)_{n\geq 1}$ be a sequence of random variables and for each integer $k \geq 1$, let $\mathcal{F}_k := \sigma(X_k, X_{k+1}, \dots)$ be the σ-field generated by $X_k, X_{k+1}, \dots$, and let $\mathcal{F}_\infty = \cap_{k=1}^{\infty} \mathcal{F}_k$.

   (a) Show that $\limsup_{n\to+\infty} X_n$ is $\mathcal{F}_\infty$-measurable.

   (b) Let now the random variables X_n, $n \geq 1$ be independent, and let $F(z) = \sum_{n=0}^{+\infty} X_n z^n$, $z \in \mathbb{C}$. Let R be the radius of convergence of the random series F. Is R almost surely constant?

3. (a) Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two sub-σ-fields of $\mathcal{G}$, and let $\mathcal{F} = \mathcal{F}_1 \vee \mathcal{F}_2$ be the σ-field generated by $\mathcal{F}_1 \cup \mathcal{F}_2$. Let Y be a random variable such that $\mathbb{E}|Y| < +\infty$ and let $\sigma(Y)$ be the σ-field generated by Y. Assuming that $\sigma(Y) \vee \mathcal{F}_1$ and $\mathcal{F}_2$ are independent, show that, almost surely,

   $$\mathbb{E}(Y|\mathcal{F}) = \mathbb{E}(Y|\mathcal{F}_1).$$

   (b) Let $(X_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables such that $\mathbb{E}|X_n| < +\infty$; and let $S_n = \sum_{k=1}^{n} X_k$. Find

   $$\mathbb{E}(X_1 | \sigma(S_n, S_{n+1}, S_{n+2} \dots)).$$

4. (a) Let $(X_n)_{n\geq 1}$ be a sequence of independent Rademacher random variables with parameter $0 < p < 1$, $p \neq 1/2$, i.e., $\mathbb{P}(X_n = -1) = 1 - p$ and $\mathbb{P}(X_n = 1) = p$, $p \neq 1/2$. Let $S_n = X_1 + X_2 + \cdots + X_n$, let $A_n = \{S_n = 0\}$, and let $S_0 = 0$. What is $\mathbb{P}(\limsup_{n\to+\infty} A_n)$ equal to?

   (b) Now consider the case of $p = 1/2$; that is, $\mathbb{P}(X_n = -1) = 1/2 = \mathbb{P}(X_n = +1)$. Prove that $\mathbb{P}(\limsup_{n\to\infty} A_n) = 1$.

   Hint. Define $A^+ = \{\limsup_{n\to\infty} S_n = +\infty\}$ and first prove that $\mathbb{P}(A^+) > 0$.

5. Let X be a nonnegative random variable and, for $x \in \mathbb{R}$, let x_+ be the positive part of x defined as $x_+ = \max\{x, 0\}$. Show that

$$\mathbb{E}(\log X)_+ < \infty \text{ if and only if } \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{P}(X > n) < \infty.$$

6. Let $(A_n)_{n \geq 1}$ be a sequence of events that are pairwise independent. Show that if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(\limsup_{n \to \infty} A_n) = 1$.

7. Let $\phi : \mathbb{R} \to \mathbb{C}$ be given by $\phi(t) = 1 - \sin^4(t)$. Is ϕ the characteristic function of a random variable? If so, compute the distribution function of the random variable. If not, prove that there is no such random variable.

8. Let $g : [0, \infty) \to [0, \infty)$ be a continuous, strictly increasing function such that $g(0) = 0$.

   1. Suppose that g is bounded. Let $(X_n)_{n \geq 1}$ be a sequence of random variables, and let X be another random variable, all defined on the same probability space. Prove that $X_n \to X$ in probability if and only if $\mathbb{E}g(|X_n - X|) \to 0$.
   2. Suppose instead that g is unbounded. Show that there exist random variables (Y_n) and Y, defined on the same probability space, such that $Y_n \to Y$ in probability but $\mathbb{E}g(|Y_n - Y|)$ does not converge to 0.