Discrete Mathematics Comprehensive Exam
Fall 2023

Student Number: ☐

Instructions: Complete **exactly 5** of the given 6 problems and **circle** their numbers below. The uncircled problems will **not** be graded.

1 2 3 4 5 6

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $\mathcal{B}$ be a basis of the cycle space of $G \cong K_{3,3}$. Show that there is an edge of G contained in at least 3 members of $\mathcal{B}$.

2. A graph G is *degree-choosable* if for every family of sets $(S_v)_{v \in V(G)}$ such that $|S_v| \geqslant d_G(v)$ for all $v \in V(G)$, G has a proper vertex-coloring c with $c(v) \in S_v$ for all $v \in V(G)$.

   Let G be a connected graph and let H be a connected induced subgraph of G. Suppose H is degree-choosable. Prove that G is also degree-choosable.

3. Let H be the graph obtained from $K_{1,3}$ by subdividing one edge. Let G be a connected graph that has no induced subgraph isomorphic to H. Let v be a vertex of G, and let G_v denote the subgraph of G induced by all vertices of distance at least 3 from v. Show that $\Delta(G_v) < R(3, \omega(G) + 1)$. (Here $\omega(G)$ is the clique number of G, and $R(s, t)$ denotes the Ramsey number for independent set of size s and clique of order t.)

4. Let G be a graph with n vertices and m edges. We say that an edge $e \in E(G)$ is ε-*light* if it belongs to at most εn triangles in G. Pick a vertex $w \in V(G)$ uniformly at random and let $U := N_G(w)$. Let X be the random variable equal to the number of ε-light edges with both endpoints in U. Show that $\mathbb{E}[X] \leqslant \varepsilon m$.

5. Let $\mathcal{F}$ be a family of subsets of $[n]$ such that $|\mathcal{F}| \leqslant n^d$, where n and d are positive integers. Show that if $|A| \leqslant n^{1/2}$ for all $A \in \mathcal{F}$, then there exists a subset $X \subseteq [n]$ of size $|X| \geqslant n^{1/3}$ such that $|X \cap A| \leqslant 10d$ for all $A \in \mathcal{F}$.

6. A 3-*term arithmetic progression*, or a 3-*AP* for short, is a set of the form $\{a, a + d, a + 2d\}$, where $a \in \mathbf{R}$ and $d > 0$. A set $S \subseteq \mathbf{R}$ is 3-*AP-free* if it does not contain a 3-AP.

   Fix a constant $c > 0$. Let $n \in \mathbf{N}$ and form a subset $S \subseteq [n]$ by including each element independently with probability $p = cn^{-2/3}$. Compute the following limit:

$$\lim_{n \to \infty} \mathbb{P}[S \text{ is 3-AP-free}].$$