Differential Equations Comprehensive Exam
Fall 2023

Student Number:

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Consider $\dot{x} = -x^{2023}$ where $x(t) \in \mathbb{R}$. Can this system admit a periodic solution? How about $\ddot{x} = -x^{2023}$? Fixed points do not count.

2. Does the initial value problem $\dot{x} = x^{1/3}, x(0) = 0, t \geq 0$ have a unique solution? If yes, prove it. If no, can you find at least three solutions and explain why standard proof of uniqueness does not apply?

3. Consider the system
$$\begin{cases} \dot{x} & = \sigma(y - x) \\ \dot{y} & = rx - y - xz \\ \dot{z} & = xy - bz \end{cases}$$
where $\sigma, r, b > 0$ are constant parameters. Can you find all fixed points in this system as functions of σ, r, b? What can you say about the stability of the fixed point $x = y = z = 0$ based on linearization? If it is unstable, does this instability guarantee the stability of (one of) the other fixed points?

4. Find the stable manifold of each fixed point in the system $\dot{x} = x - x^3$. Find all heteroclinic connections, if any.

5. Let $B(0,1)$ be the unit ball in $\mathbf{R^3}$ centered at the origin. Find a bounded solution to the following Dirichlet problem **outside** $B(0,1)$:
$$\begin{cases} -\Delta u(x) = 0, \ |x| > 1, \\ u(x) = \dfrac{2}{\sqrt{7 + 4\sqrt{3}x_3}}, \ for \ |x| = 1. \end{cases}$$

6. Consider the heat equation
$$u_t = u_{xx}, \ x \in \mathbf{R}, \ t > 0$$
with initial data $u(x,0) = g(x,0)$, where
$$g(x) = \begin{cases} 1, \ x < 0, \\ 0, \ x > 0. \end{cases}$$
For any $x \in \mathbf{R}$, show that $\lim_{t \to \infty} u(x,t) = \dfrac{1}{2}$.

7. For smooth functions $f(x,t)$, $g(x)$, and $h(x)$, prove that there is at most one solution $u \in C^2([0,1]\times[0,\infty))$ to the following problem

$$\begin{cases} u_{tt} - u_{xx} + \frac{1}{2}u = f(x,t),\ x \in (0,1),\ t > 0, \\ u(0,t) = u(1,t) = 0,\ t > 0 \\ u(x,0) = g(x),\ u_t(x,0) = h(x),\ for\ x \in (0,1), \end{cases}$$

8. Let $\Omega \subset \mathbf{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ solve

$$\Delta u(x) + \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i} u(x) = 0,\ x \in \Omega,$$

where $a_1, \cdots, a_n$ are uniformly bounded continuous functions in $\bar{\Omega}$. Prove that

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u.$$

Hint: Consider $v_\epsilon(x) = u(x) + \epsilon e^{\alpha x_1}$.