Topology Comprehensive Exam
Fall 2023

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let X and Y be the smooth submanifolds of $\mathbf{R}^3$ given by $x^2+y^2-z^2=1$ and $x+y+z=0$ respectively. Show that $X \cap Y$ is a smooth submanifold of $\mathbf{R}^3$.

2. The universal cover of any orientable surface of genus $g > 0$ is $\mathbf{R}^2$, and the fundamental group of such a surface contains no elements of positive finite order. Given these facts show that if X is a space with finite fundamental group, then any continuous map from X to a surface of genus 2 is homotopic to a constant map. On the other hand, show that there exist a space X with an infinite fundamental group, a surface Σ of genus 2, and a continuous map $f : X \to \sigma$ that is not homotopic to a constant map

3. Show that the free group F_2 on two generators contains a normal subgroup that is a free group on n generators for any $n > 2$. What is the index of this normal subgroup?

4. Let M and N be smooth manifolds. Show that if M is nonorientable, then $M \times N$ is nonorientable.

5. Show that the orthogonal group $O(n)$, which is defined as the space of $n \times n$ real matrices with orthonormal columns, is a smooth manifold and compute its dimension.

6. For each integer n let $p_n : S^1 \to S^1 : \theta \mapsto \theta + 2\pi/n$, and X be the space obtained from $S^1 \times [0,1]$ by identifying each point $(\theta, 0)$ with $(p_n(\theta), 0)$ and each point $(\theta, 1)$ with $(p_m(\theta), 1)$. Give a presentation for the fundamental group of X.

7. Let α be a closed 2-form on the 4-sphere. Show that $\alpha \wedge \alpha$ must vanish at some point.

8. Use intersection theory mod 2 to show that the torus $S^1 \times S^1$ is not simply connected.