Analysis Comprehensive Exam
Fall 2023

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $f_n : [0,1] \to \mathbb{R}$ be absolutely continuous functions, and let $f : [0,1] \to \mathbb{R}$ be measurable. Assume that $V[f_n - f] \to 0$ as $n \to \infty$. (Here $V[f]$ denotes the total variation of f over $[0,1]$).

   (a) Prove that f is absolutely continuous.

   (b) Prove that there exist constants $c_n \in \mathbb{R}$ so that the functions $g_n(x) = f_n(x) + c_n$ converge to f uniformly.

   (c) Does there necessarily exist a constant $c \in \mathbb{R}$ so that the functions $g_n(x) = f_n(x) + c$ converge to f uniformly?

2. The two parts of this question are unrelated, or related only in concept. Let $f \in L^1(\mathbb{R})$.

   (a) Prove that $f(n^2 x) \to 0$ for almost every $x \in \mathbb{R}$.

   (b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, and $f' \in L^1(\mathbb{R})$ then $f(x) \to 0$ as $|x| \to \infty$. Is the condition $f' \in L^1(\mathbb{R})$ necessarily for this conclusion?

3. Let (X,d) be a metric space, and $A \subset X$. Assume that every function $f : A \to \mathbb{R}$ that is continuous, is uniformly continuous. Show that A is closed.

4. Let X be a Banach space, Y be a normed linear space, and $B : X \times Y \mapsto \mathbb{R}$ be a bilinear function (that is, it is linear in each of its two variables). Suppose that for each $x \in X$ there exists a constant $C(x) \geq 0$ such that

$$|B(x,y)| \leq C(x)\|y\| \qquad \forall y \in Y,$$

   and for each $y \in Y$, there exists $C(y) \geq 0$ such that

$$|B(x,y)| \leq C(y)\|x\| \qquad \forall x \in X.$$

   Show that then there exists a constant $C \geq 0$ such that

$$|B(x,y)| \leq C\|x\|\|y\|$$

   for all $x \in X$ and all $y \in Y$.

5. Let $E \subset \mathbb{R}$ be a set of finite positive Lebesgue measure. Let $f_n : E \to \mathbb{R}$ for $n \geq 1$, and $f : E \to \mathbb{R}$ be Lebesgue measurable. Prove that $f_n \to f$ in measure on E iff

$$\lim_{n\to\infty} \int_E e^{-1/|f(x)-f_n(x)|} dx = 0.$$

   (We define $e^{-1/0} = 0$).

6. Assume that μ is a **finite** positive measure on X. Let f be a real valued μ–measurable function on the measure space $(X, \mathcal{M}, \mu)$. Prove that

$$\lim_{n\to\infty} \int_X \cos^{2n}(\pi f(x)) d\mu(x) = \mu\{x : f(x) \in \mathbb{Z}\} = \mu(f^{-1}(\mathbb{Z})).$$

7. Construct a function $f : [0,1] \times [0,1] \to \mathbb{R}$ such that for each $x \in [0,1]$, $f(x, \cdot)$ and $f(\cdot, x)$ are integrable over $[0,1]$, and

$$\int_0^1 \left[\int_0^1 f(x,y)\, dx\right] dy \text{ and } \int_0^1 \left[\int_0^1 f(x,y)\, dy\right] dx$$

are finite, but f is not Lebesgue integrable over $[0,1] \times [0,1]$.

8. Consider the set

$$\mathfrak{A} := \{f \in L^3(\mathbb{R}) : \int_{\mathbb{R}} |f|^2 < \infty\}.$$

Prove that $\mathfrak{A}$ is an F_σ set (that is, a countable union of closed sets) in $L^3(\mathbb{R})$.