Algebra Comprehensive Exam
Fall 2023

Student Number: [ ]

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let $R = \mathbb{Z}/1805\mathbb{Z}$. How many elements $x \in R$ satisfy $x^{19} = 1$? [Note that $1805 = 5 \cdot 19^2$.]

2. Show that every group of order $99 = 3^2 \cdot 11$ is abelian.

3. Let A be an invertible real symmetric matrix. Suppose that there exists a constant C such that for all *integer* k we have $|\operatorname{Trace}(A^k)| < C$. Show that A^2 must be the identity matrix.

4. Let A and B be principal ideal domains such that $A \subseteq B$. Suppose that p and q are relatively prime elements in A. Show that p and q are also relatively prime in B. (Recall that two elements are called *relatively prime* if the only elements dividing both are units).

5. If K is a finite extension of a field F of characteristic $p > 0$ and $\alpha \in K$ satisfies $F(\alpha) = F(\alpha^p)$, show that the minimal polynomial of α over F is separable.

6. Find the degree of a splitting field for $f(x) = x^3 - 7$ over $K = \mathbf{Q}(\sqrt{-3})$.

7. Let ζ be a primitive 37th root of unity, and let $\eta = \zeta + \zeta^{10} + \zeta^{26}$. Determine the Galois group of $\mathbf{Q}(\eta)$ over $\mathbf{Q}$.

8. Let N be an R-submodule of the R-module M. Prove that if both N and M/N are finitely generated R-modules, then so is M.