Numerical Analysis Comprehensive Exam
Spring 2022

Student Number: ☐

Instructions: Complete 5 of the 7 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Consider the following numerical integration

$$\int_a^{a+2h} f(x)dx = \frac{h}{3}f(a) + \frac{4h}{3}f(a+h) + \frac{h}{3}f(a+2h).$$

(a) What is the error term associated with this numerical integration?

(b) What is the highest order of polynomial that this numerical integration computes its integral exactly?

2. Consider solving $0 = f(r)$ for $r \in \mathrm{R}$ and $f : \mathrm{R} \to \mathrm{R}$ by Newton's method.

(a) State the Newton's formula. For $f(x) = x^2 - 4$, compute x_2 starting with $x_0 = 1$.

(b) Using the error $e_n = x_n - r$, show the local convergence analysis when $f'(r) \neq 0$. What is the order of convergence?

3. Consider the following scheme to solve $y' = f(t, y)$,

$$y_{n+1} = y_n + h(\alpha f_{n-1} + \beta f_{n-2}).$$

(a) Determine the coefficient α and β, so that this multistep method is as high order as possible. What is the order of accuracy?

(b) What can you say about the 0-stability? Is this method convergent? Why or why not?

4. Consider a differential equation $u_t + 2u_x = 0$.

(a) Write down the upwind scheme to solve this problem. What is the CFL condition?

(b) What is the stability condition? Use Fourier Transform. What is the order of the amplitude error?

5. Consider solving the following initial boundary value problem

$$\begin{aligned} &u_t = au_{xx}, \quad x \in (0,1), \quad t > 0 \\ &u(1,t) = \beta(t), \quad t \geq 0 \\ &u(0,t) = \beta(t), \quad t \geq 0 \\ &u(x,0) = f(x), \quad x \in [0,1] \end{aligned}$$

(a) Design an implicit scheme of order $\mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^2)$. Explain how to achieve this order.

(b) What is the stability condition for this method?

6. Given $A \in \mathbb{R}^{n\times n}$ a symmetric positive definite matrix. By applying the Gaussian elimination to A to eliminate its first column, one obtains the following matrix

$$\begin{bmatrix} a_{11} & a_1^T \\ 0 & A_2 \end{bmatrix},$$

where a_1^T is the first row of A except the first entry.

(a) Show that A_2 is also a symmetric positive definite matrix.

(b) If A is also a band matrix with bandwidth $2m+1$, $(m < n)$, meaning $a_{ij} = 0$ if $(i-j) > m$, R is the upper triangular matrix obtained by applying the Cholesky factorization to A. Is R a band matrix? If so, what is the bandwidth? If not, explain why.

7. Cathy considers a $(3m+1)\times 3$ matrix given in the following format:

$$\begin{bmatrix} 1 & 1 & 1 \\ \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \\ \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \\ \vdots & \vdots & \vdots \\ \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix},$$

where $m = 100$ and $\epsilon = 10^{-10}$.

(a) Can she use normal equation to find the least squares solution for $Ax = b$ on a computer with machine precision is 10^{-16}? If yes, give the algorithm to compute the solution. If not, explain the reason.

(b) If Cathy doesn't want to compute the least squares solution by normal equation, can you suggest a stable algorithm to her that can be used on this computer? You must provide steps on how to obtain the solution.