Topology Comprehensive Exam
Spring 2022

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Show there is no smooth surjection from $\mathbf{R}^1$ to $\mathbf{R}^2$.

2. Let $\omega_1, \dots, \omega_k$ be 1-forms on some open set $U \subseteq \mathbf{R}^n$ such that $(\omega_1)_p, \dots, (\omega_k)_p$ are linearly independent at each point $p \in U$. Suppose that there are 1-forms $\alpha_1, \dots, \alpha_k$ such that

$$\sum_{i=1}^{k} \omega_i \wedge \alpha_i = 0.$$

Show each of the α_i is a linear combination of the ω_i.

Hint. Recall, 1-forms $\beta_1, \dots, \beta_k$ are linearly dependent if and only if $\beta_1 \wedge \dots, \wedge \beta_k = 0$. What happens when you wedge the formula above with $\omega_1 \wedge \dots \wedge \widehat{\omega}_i \wedge \dots \wedge \omega_k$? (Here the hat means the term is omitted.)

3. Show that no two of the following spaces are homeomorphic: $\mathbf{R}, \mathbf{R}^2$, and $\mathbf{R}^3$.

4. Consider the differential form $\lambda = x_1 \wedge dx_2 + x_3 \wedge dx_4$ on $\mathbf{R}^4$.

   1. Compute $d\lambda$.
   2. Show that there is no diffeomorphism $\phi : \mathbf{R}^4 \to \mathbf{R}^4$ that satisfies $\phi^* d\lambda = d\lambda$ and that takes the unit sphere in $\mathbf{R}^4$ to a sphere of radius $r \neq 1$.

      Hint. Consider $d\lambda \wedge d\lambda$.

5. Consider the torus $T = S^1 \times S^1$.

   1. Let $C = S^1 \times \{p\}$ for some point $p \in S^1$. State the definition of intersection number mod 2 of a function $f : S^1 \to T$ with C, and show there is a map $f : S^1 \to T$ such that the intersection number of its image with C is 1.
   2. Show that the map f you found in Part 1 is not homotopic to a constant map.

6. Consider the subset X of $\mathbf{R}^2$ which is the union of the following subsets:

   - the x-axis
   - the y-axis
   - the line $y = x$
   - the semicircle given by $x^2 + y^2 = 1, x \geq 0$.

   Describe the fundamental group of X in terms of generators and relations. Draw a picture of a connected 2-fold cover of X

7. Show that any continuous map from the real projective plane $\mathbf{R}P^2$ to the torus $T = S^1 \times S^1$ must be homotopic to a constant map.

8. Let I_n be the $n \times n$ identity matrix and let $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ be a $2n \times 2n$ matrix. Define $Sp(2n)$ to be the set of all real $2n \times 2n$ matrices A that satisfy $A^T J A = J$ (here A^T is the transpose of A). Show that $Sp(2n)$ is a manifold and compute its dimension.

   Hint. consider the map $A \mapsto A^T J A$ from the set of $2n \times 2n$ matrices to the set of $2n \times 2n$ skew-symmetric matrices.