Analysis Comprehensive Exam
Spring 2022

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

NOTE:

- All functions in this exam are (extended) real-valued unless explicitly stated otherwise.
- The exterior Lebesgue measure of $E \subseteq \mathbf{R}^d$ is denoted by $|E|_e$, and if E is measurable then its Lebesgue measure is $|E|$.
- The characteristic function of a set A is denoted by χ_A.

1. Assume that $E \subseteq \mathbf{R}^d$ is measurable and f_n are measurable functions on E such that

$$\sum_{n=1}^{\infty} \big|\{|f_n| > \varepsilon\}\big| < \infty$$

for each $\varepsilon > 0$. Prove that $f_n \to 0$ pointwise a.e.

2. Assume $f_0 \in L^1[0,1]$ is nonnegative, and for each integer $n \geq 0$ define

$$f_{n+1}(x) \;=\; \left(\int_0^x f_n(t)\,dt\right)^{1/2}, \qquad \text{for } x \in [0,1].$$

Assume that $f_1(x) \leq f_0(x)$ for every $x \in [0,1]$.

(a) Prove that for each $x \in [0,1]$, the sequence $\{f_n(x)\}_{n\in\mathbf{N}}$ converges monotonically to a nonnegative number $f(x)$.

(b) Prove that f is integrable on $[0,1]$, and $f(x) = \left(\int_0^x f(t)\,dt\right)^{1/2}$ for $x \in [0,1]$.

(c) Prove that if $x \in [0,1]$ and $f(x) > 0$, then f is differentiable at x. What is $f'(x)$?

(d) Assuming $f(x) > 0$ for every $x \in (0,1]$, find an explicit simple formula for f.

3. Let X be a Banach space, and let $Y,\, Z \subseteq X$ be closed subspaces of X that satisfy

$$X \,=\, Y + Z \,=\, \{y + z : y \in Y,\, z \in Z\}$$

and $Y \cap Z = \{0\}$. Prove that there exists a constant $c > 0$ such that

$$c\left(\|y\| + \|z\|\right) \,\le\, \|y + z\| \,\le\, \|y\| + \|z\|, \qquad \text{for all } y \in Y,\, z \in Z.$$

4. Assume $f \in L^1(0,\infty)$, and define

$$g(x) = \int_0^\infty \frac{f(y)}{x+y}\,dy, \qquad x > 0.$$

Prove that g is differentiable at any point $x > 0$, and if $a > 0$ then $g' \in L^1(a,\infty)$.

5. Let E be a measurable subset of $\mathbf{R}^d$ such that $|E| < \infty$, and let f be a nonnegative, bounded function on E. Prove that if

$$\sup\left\{\int_E \phi \, : \, 0 \leq \phi \leq f, \ \phi \text{ simple}\right\} = \inf\left\{\int_E \psi \, : \, f \leq \psi, \ \psi \text{ simple}\right\},$$

then f is measurable.

6. The two parts of this problem are not related.

(a) Assume that $E \subseteq \mathbf{R}^d$ is measurable, $0 < |E| < \infty$, and $A_n \subseteq E$ are measurable sets such that $|A_n| \to |E|$ as $n \to \infty$. Prove that there exists a subsequence $\{A_{n_k}\}_{k \in \mathbf{N}}$ such that $\left|\cap A_{n_k}\right| > 0$.

(b) Given $f \in L^1(\mathbf{R})$, compute (with proof): $\displaystyle \lim_{t\to\infty} \int_{-\infty}^{\infty} |f(x-t) - f(x)|\, dx.$

Observe that this is the limit as $t \to \infty$, not $t \to 0$.

7. Suppose that $f\colon [0,1] \to \mathbf{R}$ is absolutely continuous on $[0,1]$. Prove that there exist Lipschitz functions f_n on $[0,1]$ such that $V[f - f_n] \to 0$ as $n \to \infty$, where $V[g]$ denotes the total variation of a function g over $[0,1]$.

8. Let μ_1 and μ_2 be bounded signed Borel measures on $\mathbf{R}$. Prove that if $E \subseteq \mathbf{R}$ is a Borel set, then

$$\mu(E) = \iint \chi_E(x+y)\, d\mu_1(x)\, d\mu_2(y)$$

exists, and μ defined in this way is a bounded Borel measure and $\|\mu\| \leq \|\mu_1\|\,\|\mu_2\|$.