Algebra Comprehensive Exam
Spring 2022

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

Recall for problems 1 and 2 that an element a in a ring is called nilpotent if $a^k = 0$ for some nonnegative integer k.

1. Let M be a 2×2 non-nilpotent matrix with entries in $\mathbb{C}$. Prove that there exists a complex matrix N such that $N^2 = M$. (Hint: It may be helpful to consider the Jordan Canonical Form of M.)

2. Let R be a commutative ring with identity 1.
   a. Show that the set of nilpotent elements of R is an ideal.
   b. Show that if a is nilpotent then $1 - ab$ is a unit for all $b \in R$.

3. Let G be a group of order 264 containing a subgroup H of order 33. Show that G is not simple.

4. Let G be a group of order p^n, where p is prime. Prove that for every $0 \leq k \leq n$ we have that G contains a normal subgroup of order p^k (Hint: you may want to show first that any such G has non-trivial center).

5. Suppose F is a finite field of characteristic p and order p^{mn}. Determine (with proof) the number of distinct subfields it has of order p^m.

6. Let A be a finite abelian group such that $a^{10} = 1$ for all a in A. Suppose that A has exactly 168 elements of order 10. What is the order of A?

7. An irreducible polynomial $f \in K[x]$ (K is a field) is said to be normal if for one of the roots α of the polynomial f we have that $\mathbb{Q}(\alpha)$ is a splitting field of f over K. Prove that an irreducbible $f \in \mathbb{Q}[x]$ is normal if and only if its splitting field over K has degree equal to $\deg(f)$.

8. Let $f(x) = x^3 + 4x^2 + 10x + 2$. Let K be the splitting field of f over $\mathbb{Q}$ and let G be the Galois group of K.
   a. Show that f is irreducible over $\mathbb{Q}$ and f has one real and two non-real (complex) roots.
   b. Show that 3 divides the order of G and use part a) to show that G contains an element of order 2.
   c. Determine G.