Discrete Mathematics Comprehensive Exam
Spring 2022

Student Number: ☐

Instructions: Complete **exactly 5** of the given 6 problems and **circle** their numbers below. The uncircled problems will **not** be graded.

1 2 3 4 5 6

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let G be an n-vertex graph with chromatic number $\chi(G) = r \geqslant 2$. Let $s \coloneqq n - r + 1$. Show that G is isomorphic to a subgraph of K_s^r, the complete r-partite graph with s vertices in each partition set.

2. Let G be a graph with average degree at least 24. Show that G contains a 4-connected bipartite subgraph.

3. Let G be a planar graph on n vertices. Show that there is an independent set I in G such that $|I| \geqslant n/42$ and each vertex in I has degree at most 6.

4. Fix integers $2 \leqslant k \leqslant n$ and $m \geqslant 0$. Let G be a graph with n vertices and m edges. Show that there is a k-element subset $U \subseteq V(G)$ such that the number of edges in the induced subgraph $G[U]$ satisfies
$$|E(G[U])| \geqslant \frac{k(k-1)}{n(n-1)} m.$$

5. Let $\mathcal{H}$ be a k-uniform hypergraph, where $k \geqslant 5$. Suppose that every vertex of $\mathcal{H}$ belongs to at most $\frac{1}{20k}\left(\frac{5}{4}\right)^k$ edges. Show that there is a coloring of the vertices of $\mathcal{H}$ by 5 colors such that in every edge all 5 colors are represented.

6. For the purposes of this problem, a *permutation* of length n is a sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ in which every element of the set $[n]$ appears exactly once. For a permutation σ, let $L(\sigma)$ be the largest length of an increasing subsequence of σ. For instance, $L(1, 5, 2, 4, 3, 6) = 4$, as witnessed by the subsequence $(1, 2, 4, 6)$. Suppose that a permutation σ of length n is chosen uniformly at random. Prove that
$$\mathbb{E}[L(\sigma)] = O(\sqrt{n}).$$
(In other words, there is a constant C such that $\mathbb{E}[L(\sigma)] \leqslant C\sqrt{n}$ for all large enough n.)