Probability Comprehensive Exam
Fall 2022

Student Number: 

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1  2  3  4  5  6  7  8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. Let \((A_n)\) be a sequence of (not necessarily independent) events that satisfy \(P(A_n) = 1/2\) for all \(n\). Define the upper and lower densities

\[
\overline{D} = \limsup_{n \to \infty} \frac{1}{n} \# \{k = 1, \ldots, n : A_k \text{ occurs} \}
\]

and

\[
\underline{D} = \liminf_{n \to \infty} \frac{1}{n} \# \{k = 1, \ldots, n : A_k \text{ occurs} \}
\]

Show that for any \(a < 1/2\), we have \(P(\overline{D} \geq a) > 0\), but that there exist sequences \((A_n)\) with \(\underline{D} = 0\) a.s.

2. Let \(X_1, \ldots, X_n\) be random variables that are symmetric; that is, for any \(k = 1, \ldots, n\), both \(X_k\) and \(-X_k\) have the same distribution. Show that if the \(X_k\) are independent, then the variable \(S_n = X_1 + \cdots + X_n\) is symmetric, but if the \(X_k\) are not independent, then \(S_n\) need not be symmetric.

3. Let \(X_1, X_2, \ldots\), be i.i.d. standard normal random variables. For \(n \geq 1\), set \(S_n = X_1 + \cdots + X_n\) and \(R_n = S_1 + \cdots + S_n\). Prove that

\(P(R_n \in [-1, 1] \text{ for infinitely many } n) = 0\).

4. Let \(X_1, X_2, \ldots\) be a sequence of i.i.d. random variables with \(\mathbb{E}X_k = 0\) and such that \(P(X_k \in [-1, 1] \setminus \{0\}) = 1\). Prove that

\[
\limsup_{n \to \infty} \frac{X_1 + \cdots + X_n}{\sqrt{X_1^2 + \cdots + X_n^4}} = +\infty \text{ almost surely.}
\]

5. Let \(X\) and \(Y\) be two identically distributed random variables defined on \((\Omega, \mathcal{F}, \mathbb{P})\) such that \(\mathbb{E}(X - Y)_+ < +\infty\), where \(x_+ = \max\{x, 0\}\). Show that \(\mathbb{E}(X - Y) = 0\).

6. Let \(X\) be such that \(\mathbb{E}|X| < \infty\), and let \(\mathcal{F}\) be a sigma-field. Let \(\mathcal{G}\) be a sigma-field that is independent of \(\sigma(\mathcal{F})\). Show that

\[
\mathbb{E}[X | \sigma(\mathcal{F}, \mathcal{G})] = \mathbb{E}[X | \mathcal{F}].
\]

In particular, if \(X\) is independent of \(\mathcal{G}\), then \(\mathbb{E}[X | \mathcal{G}] = \mathbb{E}X\).

7. Let \((X_n)\) be a sequence of i.i.d. random variables with exponential distribution of parameter 1. For each \(n \geq 1\), let \(M_n = \max\{X_1, \ldots, X_n\}\). Show that there exists \(c > 0\) such that

\[
\text{Var } M_n \geq c \text{ for all large } n.
\]
8. Let \((X_n)\) be a sequence i.i.d. random variables that are uniformly distributed on \(\{0, \ldots, 6\}\). Prove that \(\sum_{n=1}^{\infty} X_n 7^{-n}\) converges a.s. to a random variable with uniform distribution on \([0, 1]\).