Probability Comprehensive Exam
Fall 2022

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let (A_n) be a sequence of (not necessarily independent) events that satisfy $\mathbb{P}(A_n) = 1/2$ for all n. Define the upper and lower densities

$$\overline{\mathcal{D}} = \limsup_{n\to\infty} \frac{1}{n} \#\{k = 1, \ldots, n : A_k \text{ occurs}\} \text{ and}$$
$$\underline{\mathcal{D}} = \liminf_{n\to\infty} \frac{1}{n} \#\{k = 1, \ldots, n : A_k \text{ occurs}\}.$$

Show that for any $a < 1/2$, we have $\mathbb{P}\left(\overline{\mathcal{D}} \geq a\right) > 0$, but that there exist sequences (A_n) with $\underline{\mathcal{D}} = 0$ a.s.

2. Let $X_1, \ldots, X_n$ be random variables that are symmetric; that is, for any $k = 1, \ldots, n$, both X_k and $-X_k$ have the same distribution. Show that if the X_k are independent, then the variable $S_n = X_1 + \cdots + X_n$ is symmetric, but if the X_k are not independent, then S_n need not be symmetric.

3. Let $X_1, X_2, \ldots,$ be i.i.d. standard normal random variables. For $n \geq 1$, set $S_n = X_1 + \cdots + X_n$ and $R_n = S_1 + \cdots + S_n$. Prove that

$$\mathbb{P}(R_n \in [-1, 1] \text{ for infinitely many } n) = 0.$$

4. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with $\mathbb{E}X_k = 0$ and such that $\mathbb{P}(X_k \in [-1, 1] \setminus \{0\}) = 1$. Prove that

$$\limsup_{n\to\infty} \frac{X_1 + \cdots + X_n}{\sqrt{X_1^4 + \cdots + X_n^4}} = +\infty \text{ almost surely.}$$

5. Let X and Y be two identically distributed random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}(X - Y)_+ < +\infty$, where $x_+ = \max\{x, 0\}$. Show that $\mathbb{E}(X - Y) = 0$.

6. Let X be such that $\mathbb{E}|X| < \infty$, and let $\mathcal{F}$ be a sigma-field. Let $\mathcal{G}$ be a sigma-field that is independent of $\sigma(\sigma(X), \mathcal{F})$. Show that

$$\mathbb{E}[X \mid \sigma(\mathcal{F}, \mathcal{G})] = \mathbb{E}[X \mid \mathcal{F}].$$

In particular, if X is independent of $\mathcal{G}$, then $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}X$.

7. Let (X_n) be a sequence of i.i.d. random variables with exponential distribution of parameter 1. For each $n \geq 1$, let $M_n = \max\{X_1, \ldots, X_n\}$. Show that there exists $c > 0$ such that

$$\text{Var } M_n \geq c \text{ for all large } n.$$

8. Let (X_n) be a sequence i.i.d. random variables that are uniformly distributed on $\{0, \dots, 6\}$. Prove that $\sum_{n=1}^{\infty} X_n 7^{-n}$ converges a.s. to a random variable with uniform distribution on $[0, 1]$.