Discrete Mathematics Comprehensive Exam
Fall 2022

Student Number:

Instructions: Complete **exactly 5** of the given 6 problems and **circle** their numbers below. The uncircled problems will **not** be graded.

1 2 3 4 5 6

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let G be a plane triangulation. Show that G admits a nowhere-zero 3-flow except when $G \cong K_4$.

   Hint: You may use Tutte's theorem on flow-coloring duality for plane graphs.

2. Construct graphs G_k for $k \geqslant 1$ as follows. Start with $G_1 = K_1$, the one-vertex graph. Now suppose we have constructed G_k for some $k \geqslant 1$. Take k vertex-disjoint copies of G_k, say $G_k^1, \ldots, G_k^k$. For each sequence $v = (v_1, \ldots, v_k)$ with $v_i \in V(G_k^i)$ for all $i \in [k]$, add a new vertex x_v and edges $x_v v_i$ for each $i \in [k]$. Let G_{k+1} be the resulting graph.

   Show that the chromatic number of G_k is at least k.

3. Let G be a bipartite graph with partition sets U, V. Show that G has a matching M such that for some $M' \subseteq M$ the following holds for each $x \in U$:

   (1) there exists $y \in V$ such that $xy \in M'$, or

   (2) for every $y \in N(x)$ there exists $z \in U$ such that $yz \in M \setminus M'$.

   Hint: Start by considering the case when G has a matching M covering U.

4. Recall that a hypergraph is 2-*colorable* if it is possible to color its vertices red and blue so that no edge is monochromatic. Let H be a 3-uniform hypergraph with n vertices and n edges. Show that H has a 2-colorable induced subhypergraph on at least $3n/4$ vertices.

5. A string of symbols is called *square-free* if it does not contain the same nonempty substring twice in a row. For example, the string `abracadabra` is square-free, while `banana` is not (due to the contiguous repetition of the substring `an`).

   Show that there is a constant $C \geqslant 1$ with the following property: For any $n \in \mathbb{N}$ and any sequence L_1, …, L_n of finite sets each of which has size at least C, it is possible to pick elements $a_1 \in L_1, \dots$, $a_n \in L_n$ so that the string $a_1 \dots a_n$ is square-free.

   Hint: You may need to use that the series $\sum_{k\geqslant 1} c^k k$ is convergent for $c \in (0, 1)$.

6. Let U, V be disjoint sets of size n. Form a random bipartite graph G with partition sets U and V by adding an edge between each pair of vertices $u \in U$ and $v \in V$ independently with probability $p = p(n)$. Find a threshold function $t = t(n)$ such that

$$\lim_{n\to\infty} \mathbb{P}[\text{G contains a 4-cycle}] = \begin{cases} 0 & \text{if } p = o(t); \\ 1 & \text{if } p = \omega(t). \end{cases}$$