Student Number: 

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. Suppose that \( f : M \to N \) is a smooth map from a compact non-empty manifold \( M \) to a connected manifold \( N \). If \( df_x \) is invertible for all \( x \in M \) then show that \( f \) is surjective.

2. Let \( K \) be a submanifold of \( \mathbb{R}^3 \) diffeomorphic to the circle. For every \( \epsilon > 0 \) show there is a vector \( v \in \mathbb{R}^3 \) with length less than \( \epsilon \) such that \( K \) and \( K + v = \{ x + v : x \in K \} \) are disjoint.

3. Let \( X = \mathbb{R}P^2 \times \mathbb{R}P^2 \). Describe the universal cover of \( X \). Describe all other covers of \( X \) as quotients of the universal cover.

4. Let \( f : \mathbb{R}^3 \to \mathbb{R}^2 \) be given by \( f(x, y, z) = (x, yz) \). Let \( \omega = \sin x \, dy \) be a 1-form on \( \mathbb{R}^2 \). Compute the pull-back \( f^* \omega \), its exterior derivative \( df^* \omega \), and the Lie derivative \( L_v f^* \omega \) in the direction \( v = \frac{\partial}{\partial x} \).

5. Let \( S \) be a smooth compact connected subsurface of \( \mathbb{R}^3 \). Show there is a plane in \( \mathbb{R}^3 \) that intersects \( S \) in a non-empty union of circles. 
   Hint: Consider a projection to a coordinate axis.

6. Suppose that \( X \) and \( Y \) are homotopy equivalent CW complexes. Show that the universal covers of \( X \) and \( Y \) are homotopy equivalent.

7. Let \( S^1 = \{ (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \} \) and \( f : \mathbb{R} \to S^1 \) be the function given by \( f(t) = (\cos 2\pi t, \sin 2\pi t) \), where \( t \) is the coordinate on \( \mathbb{R} \). Show there is a unique 1-form \( \omega \) on \( S^1 \) such that \( f^* \omega = dt \). Show that \( \omega \) is closed but not exact.

8. Let \( K \) denote the Klein bottle, let \( f : \partial D^2 \to K \) be an injective map whose image is the curve \( b \) in \( K \) (see the figure). Let \( X \) be the space \( K \bigsqcup_f D^2 \), that is, the space obtained from \( K \) by attaching a disk \( D^2 \) according to \( f \). Compute the fundamental group of \( X \). Show that \( X \) is homotopy equivalent to a surface. Determine the surface.