Topology Comprehensive Exam
Fall 2022

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Suppose that $f : M \to N$ is a smooth map from a compact non-empty manifold M to a connected manifold N. If df_x is invertible for all $x \in M$ then show that f is surjective.

2. Let K be a submanifold of $\mathbb{R}^3$ diffeomorphic to the circle. For every $\epsilon > 0$ show there is a vector $v \in \mathbb{R}^3$ with length less than ϵ such that K and $K + v = \{x + v : x \in K\}$ are disjoint.

3. Let $X = \mathbf{R}P^2 \times \mathbf{R}P^2$. Describe the universal cover of X. Describe all other covers of X as quotients of the universal cover.

4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(x, y, z) = (x, yz)$. Let $\omega = \sin x\, dy$ be a 1-form on $\mathbb{R}^2$. Compute the pull-back $f^*\omega$, its exterior derivative $df^*\omega$, and the Lie derivative $L_v f^*\omega$ in the direction $v = \frac{\partial}{\partial x}$.

5. Let S be a smooth compact connected subsurface of $\mathbb{R}^3$. Show there is a plane in $\mathbb{R}^3$ that intersects S in a non-empty union of circles.
   Hint: Consider a projection to a coordinate axis.

6. Suppose that X and Y are homotopy equivalent CW complexes. Show that the universal covers of X and Y are homotopy equivalent.

7. Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and $f : \mathbb{R} \to S^1$ be the function given by $f(t) = (\cos 2\pi t, \sin 2\pi t)$, where t is the coordinate on $\mathbb{R}$. Show there is a unique 1-form ω on S^1 such that $f^*\omega = dt$. Show that ω is closed but not exact.

8. Let K denote the Klein bottle, let $f : \partial D^2 \to K$ be an injective map whose image is the curve b in K (see the figure). Let X be the space $K \coprod_f D^2$, that is, the space obtained from K by attaching a disk D^2 according to f. Compute the fundamental group of X. Show that X is homotopy equivalent to a surface. Determine the surface.