Analysis Comprehensive Exam
Fall 2022

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

NOTE:

- All scalars in this exam are real unless explicitly stated otherwise.
- All functions in this exam are (extended) real-valued unless explicitly stated otherwise.
- The exterior Lebesgue measure of $E \subseteq \mathbf{R}^d$ is denoted by $|E|_e$, and if E is measurable then its Lebesgue measure is $|E|$.
- The characteristic function of a set A is denoted by χ_A.

1. Assume that $f\colon \mathbf{R} \to [0,\infty)$ is measurable, and $\varphi\colon [0,\infty) \to [0,\infty)$ is monotonic increasing and absolutely continuous on every interval $[0,T]$ with $T > 0$. Prove that if $\varphi(0) = 0$, then
$$\int_{-\infty}^{\infty} (\varphi \circ f)(x)\, dx \;=\; \int_0^{\infty} |\{f > t\}|\, \varphi'(t)\, dt,$$
where $\{f > t\} = \{x \in \mathbf{R} : f(x) > t\}$.

2. We say that a bi-infinite sequence $\{x_n\}_{n\in\mathbf{Z}}$ in a Hilbert space H is *stationary* if $\langle x_n, x_m\rangle = \langle x_{n-m}, x_0\rangle$ for all m, $n \in \mathbf{Z}$. Prove that $\{x_n\}_{n\in\mathbf{Z}}$ is stationary if and only if there exists a unitary operator $U\colon H \to H$ such that $x_n = U^n x_0$ for $n \in \mathbf{Z}$.

3. Assume that functions $f_n \in \mathrm{AC}[a,b]$ are such that:
   - each f_n is monotone increasing on $[a,b]$,
   - $f_n(a) = 0$ for every n,
   - $\sup_n f_n(b) < \infty$, and
   - the sequence $\{f_n'(x)\}_{n\in\mathbf{N}}$ is monotone increasing for a.e. x.

   Prove that there exists an absolutely continuous function f on $[a,b]$ such that $f_n \to f$ uniformly.

4. Suppose that A and B are Lebesgue measurable subsets of $[0,1]$, and $|A| = |B| = 1/2$. Prove that there exists an $x \in [-1,1]$ such that $|(A+x)\cap B| \geq 1/10$, where $A + x = \{y + x : x \in A\}$.

5. Let $f \in L^1[0,1]$ prove that:
$$\lim_{n\to\infty} \int_0^1 f(x)\,|\sin 2\pi n x|\, dx = \frac{2}{\pi}\int_0^1 f(x)\, dx.$$
   Hint: You may want to first consider functions that have an especially simple structure.

6. Let ν be a signed measure on a measurable space (X, Σ), and let $|\nu|$ be its total variation measure. Prove that if $E \in \Sigma$, then
$$|\nu|(E) \;=\; \sup\left\{ \sum_{k=1}^{n} |\nu(E_k)| \;:\; n \in \mathbf{N},\, E_k \in \Sigma,\, E = \bigcup_{k=1}^{n} E_k \text{ disjointly} \right\}.$$

7. Let $f \in L^1(\mathbf{R})$. Show that the series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} f(x+\sqrt{k})$$

converges absolutely for almost every $x \in \mathbf{R}$.

8. Let S be the set of all nonnegative measurable functions on $[0,\infty)$ that satisfy $\int_0^\infty [f(x)]^4\,dx \leq 1$. Find

$$\sup_{f\in S} \int_0^\infty [f(x)]^3\, e^{-x}\,dx.$$