Algebra Comprehensive Exam
Fall 2022

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let G be a group of odd order and let N be a normal subgroup of G of order 5. Show that N belongs to the center of G.

2. Consider the symmetric group S_6 of order 6!. Let $\sigma = (1\,3\,5)(2\,4\,6)(1\,2\,3)(4\,5\,6)$. What is the order of σ? How many elements of S_6 are conjugate to σ? How many elements of S_6 commute with σ? Explain your answers.

3. Let R be an integral domain. Show that a polynomial of degree d in $R[x]$ has at most d roots. Give an example when R is not an integral domain and this statement fails.

4. Show that the ideal generated by x^2+1 and y^2+1 is not prime in $\mathbf{Q}[x,y]$. Find a prime ideal that contains (x^2+1, y^2+1). (Reminder: (1) is not considered prime).

5. Let A be a $n \times n$ matrix with integer entries. Show that $\mathbf{Z}$-modules $\mathbf{Z}^n/A\mathbf{Z}^n$ and $\mathbf{Z}^n/A^T\mathbf{Z}^n$ are isomorphic. Explain your solution.

6. Let X be a $n \times n$ matrix with complex entries. Consider the set V_X of $n \times n$ complex matrices commuting with X:

$$V_X = \{Y \in \mathbf{C}^{n\times n} \mid XY = YX\}.$$

   Show that V_X is a subspace of $\mathbf{C}^{n\times n}$ and $\dim V_X \geq n$.

7. Determine a splitting field of $x^4 + x^2 + 1$ over $\mathbf{Q}$. What is its degree over $\mathbf{Q}$? Is the $x^4 + x^2 + 1$ irreducible over $\mathbf{Q}$?

8. Let $K/\mathbf{Q}$ be a Galois extension of degree 9, and suppose K has at least two distinct subfields L_1, L_2 not equal to K or $\mathbf{Q}$. What is $Gal(K/\mathbf{Q})$?