Differential Equations Comprehensive Exam
Fall 2021

Student Number: [ ]

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. Find the entropy solution of
\[ u_t + (1 - u)u_x = 0 \quad \text{in } (0, \infty) \times \mathbb{R}, \]
with the initial condition
\[ u(0, x) = \begin{cases} 
0 & x < 0 \\
1 & x > 1.
\end{cases} \]

2. Let \( \Omega = \{ x \in \mathbb{R}^2 : |x| > 1 \} \) be the exterior of the unit disk in \( \mathbb{R}^2 \). Let \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) be harmonic in \( \Omega \) and bounded in \( \Omega \). Prove that
\[
\sup_{\overline{\Omega}} u = \max_{\partial \Omega} u.
\]

Give a counterexample to show that the result is false if \( \Omega \) is the exterior of the unit ball in \( \mathbb{R}^3 \).

3. Let \( u \) be the solution to the heat equation
\[
\begin{cases}
\begin{array}{ll}
\frac{\partial u}{\partial t} = \Delta u & (t, x) \in (0, +\infty) \times \mathbb{R}^d \\
u(0, x) = f(x) & x \in \mathbb{R}^d,
\end{array}
\end{cases}
\]
where \( f \in C^1(\mathbb{R}^d) \) is bounded and such that there exists \( K \geq 0 \) such that
\[
\sup_{x \in \mathbb{R}^d} |Df(x)| \leq K,
\]
where \( Du = (u_{x_1}, \ldots, u_{x_d}) \) denotes the gradient of \( f \). Show that
\[
\sup_{x \in \mathbb{R}^d} |u(t, x) - f(x)| \leq CK\sqrt{t}
\]
for some absolute constant \( C \) depending only on \( d \).

4. Let \( u \) be the solution to the wave equation
\[
\begin{cases}
\begin{array}{ll}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial x} = 0 & (t, x) \in (0, +\infty) \times \mathbb{R} \\
u(0, x) = f(x), u_t(0, x) = g(x), & x \in \mathbb{R},
\end{array}
\end{cases}
\]
where
\[
f(x) = \begin{cases}
\sin(\pi x) & 2 < x < 3 \\
0 & \text{otherwise},
\end{cases} \quad g(x) = \begin{cases}
\cos(\pi x) & 4.5 < x < 6.5 \\
0 & \text{otherwise}.
\end{cases}
\]
Find all \( t > 0 \) such that \( u(t, 1) \neq 0 \).
5. Consider the following perturbed harmonic oscillator:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x - \epsilon (y^2 - 1)y
\end{align*}
\]  

Show that for \(0 < \epsilon \ll 1\) there is one periodic orbit (not the fixed point). Compute the periodic orbit allowing errors \(O(\epsilon)\). (It is OK to leave the answer in terms of explicit integrals without computing their values).

Hint: the harmonic oscillator preserves the energy.

6. Consider the differential equation in \(\mathbb{R}^n\)

\[
\dot{y} = Ay + P(t, y)
\]

where \(A\) is a constant matrix that has only eigenvalues with negative real part.

Assume that for some \(T > 0\), we have \(P(t, y) = P(t + T, y)\) for all \(t, y\).

Assume that \(P\) has all derivatives uniformly bounded and that \(|\partial_y P(t, y)| \leq \epsilon\), with \(\epsilon\) sufficiently small.

Show that the differential equation has a periodic solution of period \(T\).

7. Consider the differential equation in \(\mathbb{R}^2\)

\[
\dot{y}(t) = \begin{pmatrix}
1 + \cos(t) & a(t) \\
b(t) + \cos(t) & 2 - \cos(t)
\end{pmatrix} y
\]

Show that for all choices of \(2\pi\) periodic functions \(a(t), b(t)\) there are solutions that grow unbounded as \(t \to \infty\).

8. Consider the differential equation in \(\mathbb{R}^2\)

\[
(\dot{x}, \dot{y}) = \left(\frac{1}{4}x, \frac{1}{2}y\right) + (0, x^2)
\]

Show that there is no twice differentiable curve in the plane going through the origin and tangent to the axis \(y = 0\) that is invariant under the differential equation.

Hint: If such a curve existed, it could be written as the graph of a function giving \(y\) as a function of \(x\).