Differential Equations Comprehensive Exam
Fall 2021

Student Number:

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Find the entropy solution of

$$u_t + (1-u)u_x = 0 \quad \text{in } (0,\infty)\times\mathbb{R},$$

with the initial condition

$$u(0,x) = \begin{cases} 0 & x<0 \\ x & 0\le x\le 1 \\ 1 & x>1. \end{cases}$$

2. Let $\Omega = \{x\in\mathbb{R}^2 : |x|>1\}$ be the exterior of the unit disk in $\mathbb{R}^2$. Let $u\in C^2(\Omega)\cap C(\overline{\Omega})$ be harmonic in Ω and bounded in $\overline{\Omega}$. Prove that

$$\sup_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Give a counterexample to show that the result is false if Ω is the exterior of the unit ball in $\mathbb{R}^3$.

3. Let u be the solution to the heat equation

$$\begin{cases} u_t = \Delta u & (t,x)\in(0,+\infty)\times\mathbb{R}^d \\ u(0,x) = f(x) & x\in\mathbb{R}^d, \end{cases}$$

where $f\in C^1(\mathbb{R}^d)$ is bounded and such that there exists $K\ge 0$ such that

$$\sup_{x\in\mathbb{R}^d} |Df(x)| \le K,$$

where $Du = (u_{x_1}, ..., u_{x_d})$ denotes the gradient of f. Show that

$$\sup_{x\in\mathbb{R}^d} |u(t,x)-f(x)| \le CK\sqrt{t}$$

for some absolute constant C depending only on d.

4. Let u be the solution to the wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0 & (t,x)\in(0,+\infty)\times\mathbb{R} \\ u(0,x) = f(x), u_t(0,x) = g(x), & x\in\mathbb{R}, \end{cases}$$

where

$$f(x) = \begin{cases} \sin(\pi x) & 2<x<3 \\ 0 & \text{otherwise} \end{cases}, \quad g(x) = \begin{cases} \cos(\pi x) & 4.5<x<6.5 \\ 0 & \text{otherwise.} \end{cases}$$

Find all $t>0$ such that $u(t,1)\neq 0$.

5. Consider the following perturbed harmonic oscillator:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - \epsilon(y^2 - 1)y \end{aligned} \tag{1}$$

   Show that for $0 < \epsilon \ll 1$ there is one periodic orbit (not the fixed point). Compute the periodic orbit allowing errors $O(\epsilon)$. (It is OK to leave the answer in terms of explicit integrals without computing their values).

   Hint: the harmonic oscillator preserves the energy.

6. Consider the differential equation in $\mathbb{R}^n$

$$\dot{y} = Ay + P(t, y)$$

   where A is a constant matrix that has only eigenvalues with negative real part.

   Assume that for some $T > 0$, we have $P(t, y) = P(t + T, y)$ for all t, y.

   Assume that P has all derivatives uniformly bounded. and that $|\partial_y P(t, y)| \leq \epsilon$, with ϵ sufficiently small.

   Show that the differential equation has a periodic solution of period T.

7. Consider the differential equation in $\mathbb{R}^2$

$$\dot{y}(t) = \begin{pmatrix} 1 + \cos(t) & a(t) \\ b(t) + \cos(t) & 2 - \cos(t) \end{pmatrix} y$$

   Show that for all choices of 2π periodic functions $a(t), b(t)$ there are solutions that grow unbounded as $t \to \infty$.

8. Consider the differential equation in $\mathbb{R}^2$

$$(\dot{x}, \dot{y}) = \left(\frac{1}{4}x, \frac{1}{2}y\right) + (0, x^2)$$

   Show that there is no twice differentiable curve in the plane going through the origin and tangent to the axis $y = 0$ that is invariant under the differential equation.

   Hint: If such a curve existed, it could be written as the graph of a function giving y as a function of x.