# Numerical Analysis
# Fall 2021

**Student Number:** ☐

*Instructions:* Complete 5 of the 7 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. [Contractive Mapping Theorem] Let $C$ be a closed subset of the real line. If $F$ is a contractive mapping of $C$ into $C$, then $F$ has a unique fixed point. Moreover, this fixed point is the limit of every sequence obtained from $x_{n+1} = F(x_n)$ with a starting point $x_o \in C$.

   Using the definition of contractive mapping, and by showing the sequence is Cauchy sequence, prove the above contractive mapping theorem.

2. (a) Write the Newton interpolating polynomial $p_3(x)$ which interpolates the function $f(x) = 2\sin(\frac{\pi}{3}x)$ at points $x = 0, 1, 2$ and 5.

   (b) (continuing (a)) what is a good upper bound for $|f(x) - p_3(x)|$ on $[0, 5]$.

   (c) Prove that if $f$ is a polynomial of degree $k$, then for $n > k$,

   $$f[x_0, x_1, \ldots, x_n] = 0.$$

   (Newton's divided difference formula).

3. Consider the ODE $y' = f(y, t)$ and let $\Delta t = t_{n+1} - t_n$. Let Scheme A be

   $$y^{n+1} = L(y^n, \Delta t)$$

   of order $r$ for the ODE, and Scheme B be defined as

   $$y^{n+1} = y^n + \frac{\Delta t}{6}\{y^n + 4L(y^n, \frac{\Delta t}{2}) + L(y^n, \Delta t)\}.$$

   What's the upper bound of $r$ so that Scheme B is $(r+1)$th order accurate? Prove your answer.

4. Consider a scheme

   $$a_1 U_{i-1}^{n+1} + a_2 U_i^{n+1} + a_3 U_{i+1}^{n+1} = b_1 U_{i-1}^n + b_2 U_i^n + b_3 U_{i+1}^n$$

   defined on a uniform grid partitioning $[0, 1]$, with initial and Dirichlet boundary conditions given. Show that it's stable in certain sense if $a_1 \le 0$, $a_2 > 0$, $a_3 \le 0$, $b_1, b_2, b_3 \ge 0$, and $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$.

5. Consider a differential equation $-u'' + u = f(x)$ for $0 < x < 1$, $u'(0) = 1$ and $u(1) = 0$. Let the space $V = \{v : v$ is a continuous function on $[0, 1], v'$ piecewise continuous and bounded on $[0, 1]$, and $v(1) = 0\}$ and $(u, w) = \int_0^1 u(x)w(x)dx$.

   (a) Find the corresponding variational problem (the weak form).

Let $V_h$ be the finite-dimensional subspace of $V$ consisting of continuous piecewise linear functions. $0 = x_1 < x_1 < \cdots < x_{M+1} = 1$. Consider the basis function of $\phi_i \in V_h$, where $\phi_j(x_i) = 1$ if $i = j$ and 0 if $i \neq j$.

(b) Draw/express a basis function $\phi_i \in V_h$ for an interior point $x_i$.

(c) Set up the finite element method: clearly describe the stiffness matrix and the load vector.

6. Assume that $\vec{x}$ is sufficiently close to an eigenvector $\vec{q}$ of a symmetric matrix $A$ with corresponding eigenvalue $\lambda$.

   (a) Show that Rayleigh quotient of $x$

$$r(x) = \frac{x^T A x}{x^T x}$$

   approximate eigenvalue $\lambda$.

   (b) Explain a good numerical algorithm to compute eigenvector $\vec{q}$ and eigenvalue $\lambda$. Give details of each steps, and give insight into why/how this algorithm work, such as convergence of the method.

7. Let $A \in \mathrm{R}^{n \times n}$, i.e. a square real matrix.

   (a) Give the definition of householder transformation $H \in \mathrm{R}^{n \times n}$, and show if it is symmetric and orthogonal. Give a geometric interpretation of the householder transformation $Hx$ for $x \in \mathrm{R}^n$.

   (b) Describe the details of how householder transformation can be used to compute QR factorization of a size 5 by 4 matrix $A$.