Topology Comprehensive Exam
Fall 2021

Student Number: [ ]

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let X be a path connected, locally path-connected space with $\pi_1(X, x_0)$ finite. Show that any two map $f, g : X \to T^n$ are homotopic, where T^n is the product of n copies of S^1.

2. Let S^1 be the unit circle in the complex plane. Consider the map $f : S^1 \times S^1 \to S^1$ given by $f(e^{i\theta}, e^{i\phi}) = e^{i(\theta+\phi)}$.

   1. Show f is a smooth map.
   2. Show the point $1 \in S^1$ is a regular value of f.
   3. What manifold is $f^{-1}(1)$?

   If you prefer not to use complex numbers then thinking of S^1 as the unit circle in $\mathbf{R}^2$ we can write $f((\cos\theta, \sin\theta), (\cos\phi, \sin\phi)) = (\cos(\theta+\phi), \sin(\theta+\phi))$ and the point $1 \in S^1 \subset \mathbf{C}$ can be thought of as $(1, 0) \in S^1 \subset \mathbf{R}^2$.

3. Let M be a smooth n-manifold and S a k-dimensional manifold. Suppose that ω is a k-form on M such that $d\omega = 0$. If f_0 and f_1 are homotopic smooth maps from S to M, show that $\int_S f_0^*\omega = \int_S f_1^*\omega$.

4. Let $\mathbf{R}^4$ have coordinates (x, y, z, w) and consider the vector fields $v = \frac{\partial}{\partial w}$ and $u = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$. Compute the Lie brackets $[v, u]$ and $[[v, u], u]$. Show that u, v, $[u, v]$, and $[[u, v], u]$ span the tangent space of $\mathbf{R}^4$ at any point.

5. Let X be a 1-dimensional submanifold of a 4–manifold W. Show that two smooth maps $f, g : S^1 \to W$ whose image is disjoint from X are homotopic in W if and only if they are homotopic in $W - X$.

6. Let D^2 be the unit disk in $\mathbf{R}^2$ and S^1 the unit circle, and $f : (\partial D^2) \to S^1 \times S^1$ be the map $f(\theta) = (3\theta, c)$ where c is some point in S^1. Set X equal to the space obtained by attaching D^2 to $S^1 \times S^1$ by the map f. Compute the fundamental group of X.

7. Let M be and n-manifold. Suppose that ω is a closed k-form on M and η is a closed l form on M. Show that $\omega \wedge \eta$ is a closed $k + l$ dimensional form and the De Rham cohomology class of $\omega \wedge \eta$ only depends on the De Rham cohomology class of ω and η.

8. Let X_1 and X_2 be connected CW complexes with base points x_1 and x_2, respectively, such that $\pi_1(X_1, x_1)$ and $\pi_1(X_2, x_2)$ are isomorphic to the finitely presented groups A and B. If W is the space formed by attaching a 1-cell to $X_1 \cup X_2$ along the base points and w_0 a point on the interior of the 1-cell, then $\pi_1(W, w_0)$ is isomorphic to the free product $A * B$. If G is an index 2 subgroup of $A * B$, show that G is isomorphic to one of the following

1. $A * A * B'$ where B' is an index 2 subgroup of B,
2. $A' * B * B$ where A' is an index 2 subgroup of A, or
3. $A' * B' * \mathbf{Z}$ where A' and B' are index 2 subgroups of A and B, respectively, and $\mathbf{Z}$ is the group of itegers.

Hint: Try to understand the covering spaces of W in terms of the covering spaces of X_1 and X_2.