Topology Comprehensive Exam
Fall 2021

Student Number: 

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. Let $X$ be a path connected, locally path-connected space with $\pi_1(X, x_0)$ finite. Show that any two maps $f, g : X \to T^n$ are homotopic, where $T^n$ is the product of $n$ copies of $S^1$.

2. Let $S^1$ be the unit circle in the complex plane. Consider the map $f : S^1 \times S^1 \to S^1$ given by $f(e^{i\theta}, e^{i\phi}) = e^{i(\theta + \phi)}$.
   1. Show $f$ is a smooth map.
   2. Show the point $1 \in S^1$ is a regular value of $f$.
   3. What manifold is $f^{-1}(1)$?

If you prefer not to use complex numbers then thinking of $S^1$ as the unit circle in $\mathbb{R}^2$ we can write $f((\cos \theta, \sin \theta), (\cos \phi, \sin \phi)) = (\cos(\theta + \phi), \sin(\theta + \phi))$ and the point $1 \in S^1 \subset \mathbb{C}$ can be thought of as $(1, 0) \in S^1 \subset \mathbb{R}^2$.

3. Let $M$ be a smooth $n$-manifold and $S$ a $k$-dimensional manifold. Suppose that $\omega$ is a $k$-form on $M$ such that $d\omega = 0$. If $f_0$ and $f_1$ are homotopic smooth maps from $S$ to $M$, show that $\int_S f_0^* \omega = \int_S f_1^* \omega$.

4. Let $\mathbb{R}^4$ have coordinates $(x, y, z, w)$ and consider the vector fields $v = \frac{\partial}{\partial w}$ and $u = \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$. Compute the Lie brackets $[v, u]$ and $[[v, u], u]$. Show that $u, v, [u, v]$, and $[[u, v], u]$ span the tangent space of $\mathbb{R}^4$ at any point.

5. Let $X$ be a 1-dimensional submanifold of a 4–manifold $W$. Show that two smooth maps $f, g : S^1 \to W$ whose image is disjoint from $X$ are homotopic in $W$ if and only if they are homotopic in $W - X$.

6. Let $D^2$ be the unit disk in $\mathbb{R}^2$ and $S^1$ the unit circle, and $f : (\partial D^2) \to S^1 \times S^1$ be the map $f(\theta) = (3\theta, c)$ where $c$ is some point in $S^1$. Set $X$ equal to the space obtained by attaching $D^2$ to $S^1 \times S^1$ by the map $f$. Compute the fundamental group of $X$.

7. Let $M$ be and $n$-manifold. Suppose that $\omega$ is a closed $k$-form on $M$ and $\eta$ is a closed $l$ form on $M$. Show that $\omega \wedge \eta$ is a closed $k + l$ dimensional form and the De Rham cohomology class of $\omega \wedge \eta$ only depends on the De Rham cohomology class of $\omega$ and $\eta$.

8. Let $X_1$ and $X_2$ be connected CW complexes with base points $x_1$ and $x_2$, respectively, such that $\pi_1(X_1, x_1)$ and $\pi_1(X_2, x_2)$ are isomorphic to the finitely presented groups $A$ and $B$. If $W$ is the space formed by attaching a 1-cell to $X_1 \cup X_2$ along the base points and $w_0$ a point on the interior of the 1-cell, then $\pi_1(W, w_0)$ is isomorphic to the free product $A * B$. If $G$ is an index 2 subgroup of $A * B$, show that $G$ is isomorphic to one of the following.
1. \( A \ast A \ast B' \) where \( B' \) is an index 2 subgroup of \( B \),
2. \( A' \ast B \ast B \) where \( A' \) is an index 2 subgroup of \( A \), or
3. \( A' \ast B' \ast Z \) where \( A' \) and \( B' \) are index 2 subgroups of \( A \) and \( B \), respectively, and \( Z \) is the group of integers.

Hint: Try to understand the covering spaces of \( W \) in terms of the covering spaces of \( X_1 \) and \( X_2 \).