Probability Comprehensive Exam
Fall 2021

Student Number: 

Instructions: Complete 5 of the 8 problems, and circle their numbers below – the uncircled problems will not be graded.

1 2 3 4 5 6 7 8

Write only on the front side of the solution pages. A complete solution of a problem is preferable to partial progress on several problems.
1. Let the probability space $(\Omega, \Sigma, \mathbb{P})$ be the interval $(0, 1]$ together with the $\sigma$–field of Borel subsets of $(0, 1]$ and $\mathbb{P}$ being the usual Lebesgue measure on $(0, 1]$. Define the random variable $X : \Omega \to \mathbb{R}$ via the relations

$$X(2^{-j} + 2^{-j}t) := t, \quad j = 1, 2, \ldots; \quad t \in (0, 1].$$

Note that $X$ is uniformly distributed on $[0, 1]$. Show that

(a) For every $p \in (0, 1)$, there exists a Bernoulli($p$) random variable on $(\Omega, \Sigma, \mathbb{P})$ independent from $X$;

(b) Whenever $Y$ is a random variable on $(\Omega, \Sigma, \mathbb{P})$ uniformly distributed on the set \{1, 2, 3\}, $Y$ is not independent from $X$.

2. Define a set

$$T := [0, 1] \setminus \bigcup_{j=1}^{\infty}[2^{-j}, 2^{-j} + 4^{-j}].$$

Let $X$ be a random variable uniformly distributed on $T$. Compute the mean of $X$. Simplify the answer whenever possible.

3. For each $n \in \mathbb{N}$, let $b_{1n}, b_{2n}, \ldots, b_{nn}$ be Bernoulli(1/$n$) random variables, and assume that $\{b_{ij}\}_{i \leq j}$ are mutually independent. Further, let $\{g_{ij}\}_{i \leq j}$ be the triangular array of i.i.d standard Gaussian variables, and assume that $\{b_{ij}\}_{i \leq j}$ and $\{g_{ij}\}_{i \leq j}$ are mutually independent. For each $n$, we define a random variable

$$\xi_n := \sum_{i=1}^{n} b_{in} g_{in}.$$ 

Prove that the sequence $(\xi_n)_{n=1}^{\infty}$ converges in distribution to a product $\sqrt{\eta} g$, where $g$ is a standard Gaussian, $\eta$ is Poisson variable with parameter 1, and $\eta$ and $g$ are independent.

4. For every $n \in \mathbb{N}$, let $S_{n1}, S_{n2}, \ldots$ be i.i.d set-valued random variables, where each $S_{ni}$, $i = 1, 2, \ldots, n$, is uniformly distributed on the collection of all $n$–subsets of $\{1, 2, \ldots, 2n\}$. For every $n$, we define an integer valued random variable

$$\xi_n := \inf \left\{ i \geq 1 : \bigcup_{j=1}^{i} S_{nj} = \{1, 2, \ldots, 2n\} \right\}.$$ 

Prove that the sequence $(\frac{\xi_n}{\log_2(2n)})_{n=1}^{\infty}$ converges to 1 in probability.

5. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $L^0$ be the corresponding set of real-valued random variables (a.s. defined) on $(\Omega, \Sigma, \mathbb{P})$. For $X, Y \in L^0$, let $d(X, Y) = \mathbb{E}(\arctan |X - Y|)$. 

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(a) Show that $d$ defines a metric on $L^0$.

(b) Show that a sequence of random variables $(X_n)_{n \geq 1}$ converges in probability to $X_\infty$ if and only if $\lim_{n \to +\infty} d(X_n, X_\infty) = 0$.

6. Let $X$ be a random variable, with finite second moment, and whose law is absolutely continuous (with respect to Lebesgue measure) and has bounded density $f_X$ and let $f_{\text{max}} = \sup_{x \in \mathbb{R}} f_X(x)$.

(a) Show that

$$\text{Var}X \geq \frac{1}{12f_{\text{max}}^2}.$$

**Hint:** Show that the function given by $u(x) := \mathbb{P}(|X - EX| \geq x)$ is Lipschitz and more precisely that for all $x, y \in \mathbb{R}$, $|u(x) - u(y)| \leq 2\|f_X\|_{\infty}|x - y|$.

(b) Now, let $Y$ be a discrete random variable and let $p_{\text{max}} = \max_y \mathbb{P}(Y = y)$. Show that

$$\text{Var}Y \geq \frac{1}{12} \left( \frac{1}{p_{\text{max}}^2} - 1 \right).$$

**Hint:** Consider $X = Y + U$, where $U$ is uniformly distributed on $(-1/2, 1/2)$ and is independent of $Y$.

7. Let $F$ the distribution function, associated with some probability measure $\mathbb{P}$, let $(X_n)_{n \geq 1}$ be a sequence of i.i.d. (independent identically distributed) random variables having distribution function $F$, and let $Y_n = \min\{X_1, \ldots, X_n\}$.

(a) Let $F$ be given by $F(x) = 0$, for $x < a$; $0 < F(x) < 1$, for $a < x < b$; and $F(x) = 1$, for $x > b$; where $a, b \in \mathbb{R}, a < b$. Does $(Y_n)_{n \geq 1}$ converge in distribution? If yes, what is the limiting law? Justify your answer.

(b) Let now $(X_n)_{n \geq 1}$, be uniformly distributed on $[0, 1]$, does $(nY_n)_{n \geq 1}$ converge in distribution? If yes, what is the limiting law? Justify your answer.

8. Let $(X_n)_{n \geq 1}$ be a sequence of independent random variables each with finite first moment and such that for all $n$, $EX_n = 0$. Show that if

$$\sum_{n=1}^{+\infty} \mathbb{E} \left( \frac{X_n^2}{1 + |X_n|} \right) < +\infty,$$

then the series $\sum_{n=1}^{+\infty} X_n$ converges with probability one.