Probability Comprehensive Exam
Fall 2021

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let the probability space $(\Omega, \Sigma, \mathbb{P})$ be the interval $(0,1]$ together with the σ–field of Borel subsets of $(0,1]$ and $\mathbb{P}$ being the usual Lebesgue measure on $(0,1]$. Define the random variable $X : \Omega \to \mathbf{R}$ via the relations

$$X(2^{-j} + 2^{-j}t) := t, \quad j = 1, 2, \ldots; \;\; t \in (0,1].$$

Note that X is uniformly distributed on $[0,1]$. Show that

   (a) For every $p \in (0,1)$, there exists a Bernoulli(p) random variable on $(\Omega, \Sigma, \mathbb{P})$ independent from X;

   (b) Whenever Y is a random variable on $(\Omega, \Sigma, \mathbb{P})$ uniformly distributed on the set $\{1, 2, 3\}$, Y is **not** independent from X.

2. Define a set

$$T := [0,1] \setminus \bigcup_{j=1}^{\infty} [2^{-j}, 2^{-j} + 4^{-j}].$$

Let X be a random variable uniformly distributed on T. Compute the mean of X. Simplify the answer whenever possible.

3. For each $n \in \mathbf{N}$, let $b_{1n}, b_{2n}, \ldots, b_{nn}$ be Bernoulli$(1/n)$ random variables, and assume that $\{b_{ij}\}_{i \le j}$ are mutually independent. Further, let $\{g_{ij}\}_{i \le j}$ be the triangular array of i.i.d standard Gaussian variables, and assume that $\{b_{ij}\}_{i \le j}$ and $\{g_{ij}\}_{i \le j}$ are mutually independent. For each n, we define a random variable

$$\xi_n := \sum_{i=1}^{n} b_{in}\, g_{in}.$$

Prove that the sequence $(\xi_n)_{n=1}^{\infty}$ converges in distribution to a product $\sqrt{\eta}\, g$, where g is a standard Gaussian, η is Poisson variable with parameter 1, and η and g are independent.

4. For every $n \in \mathbf{N}$, let $S_{n1}, S_{n2}, \ldots$ be i.i.d set-valued random variables, where each S_{ni}, $i = 1, 2, \ldots$, is uniformly distributed on the collection of all n–subsets of $\{1, 2, \ldots, 2n\}$. For every n, we define an integer valued random variable

$$\xi_n := \inf\Big\{ i \ge 1 : \bigcup_{j=1}^{i} S_{nj} = \{1, 2, \ldots, 2n\} \Big\}.$$

Prove that the sequence $\big(\frac{\xi_n}{\log_2(2n)}\big)_{n=1}^{\infty}$ converges to 1 in probability.

5. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let L^0 be the corresponding set of real-valued random variables (a.s. defined) on $(\Omega, \Sigma, \mathbb{P})$. For $X, Y \in L^0$, let $d(X,Y) = \mathbb{E}(\arctan |X - Y|)$.

(a) Show that d defines a metric on L^0.

(b) Show that a sequence of random variables $(X_n)_{n\geq 1}$ converges in probability to X_∞ if and only if $\lim_{n\to+\infty} d(X_n, X_\infty) = 0$.

6. Let X be a random variable, with finite second moment, and whose law is absolutely continuous (with respect to Lebesgue measure) and has bounded density f_X and let $f_{\max} = \sup_{x\in\mathbf{R}} f_X(x)$.

(a) Show that

$$\mathbf{Var}X \geq \frac{1}{12 f_{\max}^2}.$$

Hint: Show that the function given by $u(x) := \mathbb{P}(|X - EX| \geq x)$ is Lipschitz and more precisely that for all $x, y \in \mathbf{R}$, $|u(x) - u(y)| \leq 2\|f_X\|_\infty |x - y|$.

(b) Now, let Y be a discrete random variable and let $p_{\max} = \max_y \mathbb{P}(Y = y)$. Show that

$$\mathbf{Var}Y \geq \frac{1}{12}\left(\frac{1}{p_{\max}^2} - 1\right).$$

Hint: Consider $X = Y + U$, where U is uniformly distributed on $(-1/2, 1/2)$ and is independent of Y.

7. Let F the distribution function, associated with some probability measure $\mathbb{P}$, let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. (independent identically distributed) random variables having distribution function F, and let $Y_n = \min\{X_1, \dots, X_n\}$.

(a) Let F be given by $F(x) = 0$, for $x < a$; $0 < F(x) < 1$, for $a < x < b$; and $F(x) = 1$, for $x > b$; where $a, b \in \mathbf{R}, a < b$. Does $(Y_n)_{n\geq 1}$ converge in distribution? If yes, what is the limiting law? Justify your answer.

(b) Let now $(X_n)_{n\geq 1}$, be uniformly distributed on $[0, 1]$, does $(nY_n)_{n\geq 1}$ converge in distribution? If yes, what is the limiting law? Justify your answer.

8. Let $(X_n)_{n\geq 1}$ be a sequence of independent random variables each with finite first moment and such that for all n, $\mathbb{E}X_n = 0$. Show that if

$$\sum_{n=1}^{+\infty} \mathbb{E}\left(\frac{X_n^2}{1 + |X_n|}\right) < +\infty,$$

then the series $\sum_{n=1}^{+\infty} X_n$ converges with probability one.