Analysis Comprehensive Exam
Fall 2021

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

NOTE:

- All functions in this exam are (extended) real-valued.
- The exterior Lebesgue measure of $E \subseteq \mathbf{R}^d$ is denoted by $|E|_e$, and if E is measurable then its Lebesgue measure is $|E|$.

1. (a) Let $f\colon (0,1] \to [0,\infty)$ be a nonnegative continuous function whose improper Riemann integral

$$I = \lim_{a\to 0^+} \int_a^1 f(x)\,dx$$

exists and is finite. Prove that f is Lebesgue integrable over $[0,1]$ and its Lebesgue integral over $[0,1]$ equals I.

(b) Does the conclusion of part (a) still hold if f is not assumed to be nonnegative?

2. Prove that if $f \in L^1(\mathbf{R})$, then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\,\sin nx\,dx = 0.$$

Remark: Give a direct proof, do not appeal to any form of the Riemann–Lebesgue Lemma.

Hint: First prove the result for step functions (finite linear combinations of characteristic functions of intervals, also known as really simple functions).

3. Let $1 \le p,\, q < \infty$, and consider the vector spaces

$$\begin{aligned} L^p(\mathbf{R}) \cap L^q(\mathbf{R}) &= \big\{f : f \in L^p(\mathbf{R}) \text{ and } f \in L^q(\mathbf{R})\big\}, \\ L^p(\mathbf{R}) + L^q(\mathbf{R}) &= \big\{f + g : f \in L^p(\mathbf{R}) \text{ and } g \in L^q(\mathbf{R})\big\}. \end{aligned}$$

As usual, we identify any two function that are equal almost everywhere.

(a) Prove that $\|f\|_{p\cap q} = \|f\|_p + \|f\|_q$ defines a norm on $L^p(\mathbf{R}) \cap L^q(\mathbf{R})$, and that $L^p(\mathbf{R}) \cap L^q(\mathbf{R})$ is a Banach space with respect to this norm.

(b) Prove that if $1 \le p < r < q < \infty$, then

$$L^p(\mathbf{R}) \cap L^q(\mathbf{R}) \subseteq L^r(\mathbf{R}) \subseteq L^p(\mathbf{R}) + L^q(\mathbf{R}).$$

4. The two parts of this problem are not related.

(a) Let $f\colon \mathbf{R} \to \mathbf{R}$ be measurable. Prove that

$$B = \big\{(x,y) \in \mathbf{R}^2 : f(x) \ge f(y)\big\}$$

is a measurable subset of $\mathbf{R}^2$.

(b) For each $n \in \mathbf{N}$, assume that E_n is a measurable subset of $[0,1]$ with $|E_n| \ge 1/2$. Let E be the set of all x in $[0,1]$ that belong to infinitely many of the sets E_n. Prove that $|E| \ge 1/2$.

5. Let U be a σ-algebra of subsets of a nonempty set X. Suppose that $\mu\colon U \to [0,\infty]$ is finitely additive on U, and $\mu(\varnothing) = 0$ while $\mu(X) < \infty$. Prove that the following two statements are equivalent.

   (a) μ is countably additive on U.

   (b) For each decreasing sequence $A_1 \supseteq A_2 \supseteq \cdots$ of subsets of U that satisfy $\cap_n A_n = \varnothing$, we have $\lim_{n\to\infty} \mu(A_n) = 0$.

   Remark: We are not assuming that μ satisfies countable subadditivity.

6. Assume that $\{e_n(x)\}_{n\in\mathbf{N}}$ is an orthonormal basis for $L^2[0,1]$, and let $g \in L^\infty[0,1]$. Prove that if $\{g(x)\, e_n(x)\}_{n\in\mathbf{N}}$ is an orthonormal basis for $L^2[0,1]$, then $|g(x)| = 1$ a.e. on $[0,1]$.

7. The two parts of this question are not related.

   (a) Let S be a subspace of a normed space X. Prove that the closure of S satisfies

$$\overline{S} \;=\; \bigcap \bigl\{\ker(\mu) : \mu \in X^* \text{ and } S \subseteq \ker(\mu)\bigr\}.$$

   (b) Let X and Y be Banach spaces, and assume that $A\colon X \to Y$ is linear. Prove that if $\mu \circ A \in X^*$ for every $\mu \in Y^*$, then A is continuous.

8. Assume that $f\colon \mathbf{R} \to \mathbf{R}$ is continuous, and $g\colon [a,b] \to [0,\infty)$ is absolutely continuous. Prove that

$$G(x) \;=\; \int_0^{g(x)} f(t)\, dt$$

   is absolutely continuous on $[a,b]$, and find G'.