Algebra Comprehensive Exam
Fall 2021

Student Number: ☐

Instructions: Complete 5 of the 8 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6 7 8

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let G be a p-group, so $|G| = p^k$ for some prime p. Let H be a normal subgroup of G of order p. Show that H lies in the center of G.

2. Let M be the quotient abelian group $\mathbb{Z}^4/A$ where A is the subgroup of $\mathbb{Z}^4$ generated by $(1, 1, 1, 1)$, $(0, 1, 1, 0)$ and $(1, 2, -8, 1)$.
   1. Write M as a direct sum of cyclic groups.
   2. How many homomorphisms are there from M to $\mathbb{Z}_4$?

3. Let A be a square matrix with entries in $\mathbb{C}$ such that A^{2020} is diagonalizable. Show that A^{2021} is also diagonalizable.

4. Let R be an integral domain. Recall that if M is an R-module, the rank of M is defined to be the maximum number of R-linearly independent elements of M.
   (a) Prove that for any R-module M, the rank of $\mathrm{Tor}(M)$ is 0.
   (b) Prove that the rank of M equals the rank of $M/\mathrm{Tor}(M)$.
   (c) Suppose that M is a non-principal ideal of R and consider it as an R-module. Prove that M is torsion-free of rank 1 but not free.

5. Let R be the (noncommutative) ring of 3×3 matrices over $\mathbb{Q}$, and S denote the ring of 2×2 matrices over $\mathbb{Q}$. Show that there is no non-trivial homomorphism from R to S.

6. Let R be a commutative ring without an identity. Prove that for all $x \in R$, the ideal xR is proper.

7. Let $F \subset K \subset L$ be finite degree field extensions. For each of the following assertions, give a proof or a counterexample.
   (a) If L/F is Galois, then so is L/K.
   (b) If K/F and L/K are both Galois, then so is L/F.

8. Let $f(x) = x^3 + 2x^2 + 6x + 10$. Let E be the splitting filed of f over $\mathbb{Q}$.
   (a) Find the degree of extension $E : \mathbb{Q}$.
   (b) Find the Galois group $\mathrm{Gal}(E : \mathbb{Q})$.