Discrete Mathematics Comprehensive Exam
Fall 2021

Student Number: ☐

Instructions: Complete 5 of the 6 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let G be a connected graph and $x, y, z \in V(G)$ be distinct. Prove that there exists a vertex $v \in V(G)$ and three paths P_x, P_y, P_z from v to x, y, z, respectively, such that $V(P_x \cap P_y) = V(P_y \cap P_z) = V(P_z \cap P_x) = \{v\}$.

2. Let $r > 0$ and let G be a graph with maximum degree at most r. Suppose for every $T \subseteq V(G)$ with $|T|$ odd, the number of edges of G with exactly one vertex in T is at least r. Show that G has a perfect matching.

3. Let G be a triangle-free planar graph. Prove that the minimum degree of G is at most 3. Use this assertion to prove that G is also 4-colorable.

4. Show that for all $n \geq 2$ there exists an n-vertex K_4-free graph G_n with at least $e(G_n) \geq cn^{8/5}$ many edges, where $c > 0$ is some constant (i.e., which does not depend on n).

5. Let M be a $n \times n$, 0-1 matrix chosen uniformly at random from the set of all such matrices. Show that if $2\log_2 n \leq k \leq n$, then the probability that M has a $k \times k$ sub-matrix of all 1s goes to zero as $n \to \infty$.

6. Suppose that we throw m balls into n bins independently and uniformly at random (initially all bins are empty, of course). Prove that $m^*(n) = n \log n$ is a sharp threshold function for the property 'there exists an empty bin', i.e., that for any $\epsilon > 0$ we have

$$\mathbb{P}(\text{there exists an empty bin}) \to \begin{cases} 1 & m \leq (1-\epsilon)n \log n, \\ 0 & m \geq (1+\epsilon)n \log n. \end{cases}$$

Hint: *Recall that* $1 - x = e^{-x+O(x^2)}$ *as* $x \to 0$.