Discrete Mathematics Comprehensive Exam
Spring 2021

Student Number:

Instructions: Complete 5 of the 6 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6

Write **only on the front side** of the solution pages. A **complete solution** of a problem is preferable to partial progress on several problems.

1. Let G be a 2-connected graph with at least 4 vertices. Show that there exists an edge e such that G/e, the graph obtained from G by contracting e, is still 2-connected.

2. Let G be a graph and d a positive integer. Suppose $e(H) < d|H|$ for all subgraphs H of G. Show that G is $(2d)$-colorable.

3. Let G be a connected, triangle-free graph and assume that no four vertices of G induce a subgraph whose edge set is a matching of size 2 in G. Prove that G is 3-colorable. (Hint: Consider an induced odd cycle in G.)

4. Show that there exists a 2-coloring of the edges of the complete n-vertex graph K_n that contains at most $\frac{1}{4}\binom{n}{3}$ many monochromatic triangles.

5. The diameter $\mathrm{diam}(G)$ of a graph G is the maximum distance between two vertices of G, where distance is the length of shortest path (so only the complete graph has diameter one). Let $G(n,p)$ denote the binomial random graph with vertex set $[n]$, i.e., where each of the $\binom{n}{2}$ possible edges is inserted independently with probability p. For constant edge-probability $p \in (0,1)$, show that $\mathbb{P}(\mathrm{diam}(G(n,p)) = 2) \to 1$ as $n \to \infty$.

6. As before, by $G(n,p)$ we denote the binomial random graph with vertex set $[n]$. Let $\omega = \omega(n) \to \infty$ as $n \to \infty$. For any $p = p(n) \in [0,1]$ show that

$$\mathbb{P}(G(n,p) \text{ is a forest}) = \begin{cases} 1 - o(1) & \text{if } p \le \omega^{-1} n^{-1}, \\ o(1) & \text{if } p \ge \omega n^{-1}, \end{cases}$$

providing full/complete details for your reasoning.