Discrete Mathematics Comprehensive Exam
Fall 2019

Student Number: [ ]

Instructions: Complete 5 of the 6 problems, and **circle** their numbers below – the uncircled problems will **not** be graded.

1 2 3 4 5 6

Write **only on the front side** of the solution pages. A student will pass the exam if 3 problems are worked "almost perfectly" and some progress is made on a fourth problem.

1. Let G be a bipartite graph with partition sets V_1, V_2 such that $|V_1| = |V_2| = n$. Find a matrix such that the permanent of that matrix counts the number of perfect matchings in G. (Provide your justification.)

2. Prove that for every bipartite graph H there exists a $\Delta(H)$-regular bipartite graph G such that $H \subseteq G$. (Multiple edges are allowed.)

3. Let G be a connected graph with minimum degree $k \geq 2$. Prove that there is a path P in G on k vertices such that $G - P$ is connected. (Hint: Use a DFS tree.)

4. Show that it is possible to color the edges of K_n with at most $3\sqrt{n}$ colors so that there are no monochromatic triangles.

5. Given a family $\mathcal{F}$ of subsets of $[n]$ and $A \subseteq [n]$, write $\mathcal{F}|_A := \{S \cap A : S \in \mathcal{F}\}$ (its projection onto A). Prove that for every n and k, there exists a family $\mathcal{F}$ of subsets of $[n]$ with $|\mathcal{F}| = O(k2^k \log n)$ such that for every k-element subset A of $[n]$, $\mathcal{F}|_A$ contains all 2^k subsets of A.

6. Let $G = (V, E)$ be a graph. Color every edge *red* or *blue* independently and uniformly at random. Let E_0 be the set of red edges and E_1 the set of blue edges. Let $G_i = (V, E_i)$ for each $i = 0, 1$. Prove or disprove:

$$\mathbb{P}(G_0 \text{ and } G_1 \text{ are both connected}) \leq \mathbb{P}(G_0 \text{ is connected})^2 .$$

   Hint: Randomly coloring the graph may be seen as choosing a random vertex of the hypercube of dimension $m = |E|$, the number of edges.