Math 1553 Worksheet §3.4-3.6

Solutions

1. True or false. Answer true if the statement is *always* true. Otherwise, answer false. If your answer is false, either give an example that shows it is false or (in the case of an incorrect formula) state the correct formula.

   a) If A is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each b in $\mathbf{R}^n$, then A must be invertible.

   b) If A is a 3×4 matrix and B is a 4×2 matrix, then the linear transformation Z defined by $Z(x) = ABx$ has domain $\mathbf{R}^3$ and codomain $\mathbf{R}^2$.

   ## Solution.

   a) True. This is part of the Invertible Matrix Theorem. We could also just count pivots: if $Ax = b$ is consistent for all b in $\mathbf{R}^n$, then A has a pivot in each of its n rows, so A has a pivot in each of its n columns. Therefore, $T(x) = Ax$ is a one-to-one and onto linear transformation, and A is invertible.

   b) False. In order for Bx to make sense, x must be in $\mathbf{R}^2$, and so Bx is in $\mathbf{R}^4$ and $A(Bx)$ is in $\mathbf{R}^3$. Therefore, the domain of Z is $\mathbf{R}^2$ and the codomain of Z is $\mathbf{R}^3$.

2. A is $m \times n$ matrix, B is $n \times m$ matrix. Select all correct answers from the box. It is possible to have more than one correct answer.

   a) Suppose x is in $\mathbf{R}^m$. Then ABx *must be* in:

      | Col(A), Nul(A), Col(B), Nul(B) |

   b) Suppose x in $\mathbf{R}^n$. Then BAx *must be* in:

      | Col(A), Nul(A), Col(B), Nul(B) |

   c) If $m > n$, then columns of AB could be linearly | *independent*, *dependent* |

   ## Solution.
   Recall, AB can be computed as A multiplying every column of B. That is $AB = \begin{pmatrix} Ab_1 & Ab_2 & \cdots Ab_m \end{pmatrix}$ where $B = \begin{pmatrix} b_1 & b_2 & \cdots b_m \end{pmatrix}$.

   a) | Col(A) |. Denote $w := Bx$, which is a vector in $\mathbf{R}^n$. $ABx = A(Bx)$ is multiplying A with w which will end up with "linear combination of columns of A". So ABx is in Col(A).

   b) | Col(B) |. Similarly, $BAx = B(Ax)$ is multiplying B with Ax, a vector in R^m, which will end up with "linear combination of columns of B". So BAx is in Col(B).

   c) | *dependent* |. Since $m > n$ means A matrix can have at most n pivots. So $dim(\text{Col}(A)) \leq n$. Notice from first question we know Col(AB) $\subset$ Col(A) which

has dimension at most n. That means AB can have at most n pivots. But AB is $m \times m$ matrix, then columns of AB must be dependent.

To summarize what we are actually study here, there are several relations between these subspaces. The symbol $\subseteq$ means "is contained in (or possibly equal to)..."

$$\text{Col}(AB) \subseteq \text{Col}(A);$$
$$\text{Col}(BA) \subseteq \text{Col}(B);$$
$$\text{Nul}(A) \subseteq \text{Nul}(BA);$$
$$\text{Nul}(B) \subseteq \text{Nul}(AB);$$

3. Consider the following linear transformations:

$T: \mathbf{R}^3 \longrightarrow \mathbf{R}^2$ T projects onto the xy-plane, forgetting the z-coordinate

$U: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ U rotates clockwise by $90°$

$V: \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ V scales the x-direction by a factor of 2.

Let A, B, C be the matrices for T, U, V, respectively.
  a) Write A, B, and C.

  b) Compute the matrix for $U \circ V \circ T$.

  c) Describe U^{-1} and V^{-1}, and compute their matrices.

Solution.

  a) To compute A, we plug in the unit coordinate vectors:

  $$T(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad T(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad T(e_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \implies \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

  Also, it is straightforward that $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

  b) $BCA = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}.$

  c) Intuitively, if we wish to "undo" U, we can imagine that we have rotated a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ by $90°$ clockwise and we want to return the vector back to its original position of $\begin{pmatrix} x \\ y \end{pmatrix}$. To do this, we need to rotate it $90°$ *counterclockwise*. Therefore, U^{-1} is counterclockwise rotation by $90°$.

  Similarly, to undo the transformation V that scales the x-direction by 2, we need to scale the x-direction by $1/2$, so V^{-1} scales the x-direction by a factor of $1/2$.

Their matrices are, respectively,

$$B^{-1} = \frac{1}{0 \cdot 0 - (-1) \cdot 1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$C^{-1} = \frac{1}{2 \cdot 1 - 0 \cdot 0} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}.$$