Math 1553 Worksheet §2.3, S2.4

Solutions

1. True or false. If the statement is *always* true, answer true. Otherwise, answer false and give an example to show it is false.

a) Suppose A is an $m \times n$ matrix and b is a vector in $\mathbf{R}^m$. If $Ax = b$ is inconsistent, then A does not have a pivot in every column.

b) Suppose A is a 3×3 matrix with two pivots, and suppose that b is a vector so that $Ax = b$ is consistent. Then the solution set for $Ax = b$ is a plane.

Solution.

a) False, for instance consider

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

This is an inconsistent system even though A has a pivot in each column.

b) False. The matrix A has two pivots, which means that the *column span* of A is plane, but this is not what the question is asking! It is asking about the solution set to $Ax = b$, not the column span of A.

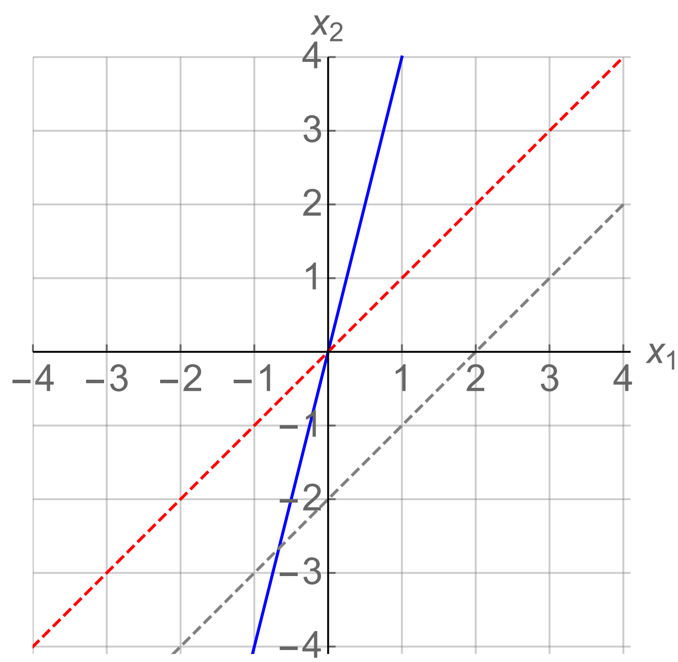
Since $Ax = b$ corresponds to a system of 3 equations in 3 variables, the fact that A has two pivots means that the system will have exactly one free variable, so the solution set will be a line in $\mathbf{R}^3$.

2. Let $A = \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}$. On the same graph, draw each of the following:
(a) The span of the columns of A.

(b) The set of solutions to $Ax = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(c) The set of solutions to $Ax = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$.
Label each of these clearly.

Solution.



The blue line is the span of columns of A: Span$\left\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right\}$. If you draw the two column vectors, you will see they both fall on the line $x_2 = 4x_1$.

The red dashed line is the span of solutions of $Ax = 0$: Span$\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$. To see this is the case, you can row reduce the augmented matrix to RREF, which is $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$. That implies the solution set is the line $x_2 = x_1$.

The gray dashed line is the set of solutions to $Ax = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$. To see this is the case, you can row reduce the corresponding augmented matrix to RREF, which is $\left(\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right)$. That implies the solution set is the line $x_1 = 2 + x_2$ (where x_2 is free) which yields parametric vector form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 + x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In other words, this solution set is the line through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ parallel to the span of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

3. Find the set of solutions (x_1, x_2, x_3) to $x_1 - 3x_2 + 5x_3 = 3$ and write the solutions in parametric vector form. Describe the solution set geometrically.

Solution.

The system $x_1 - 3x_2 + 5x_3 = 3$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 3 \end{array}\right)$ which has two free variables x_2 and x_3.

$$x_1 = 3 + 3x_2 - 5x_3 \qquad x_2 = x_2 \qquad x_3 = x_3.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 + 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

This solution set (red) is the *translation by* $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ of the plane (green) spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.

Note that $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$ both solve the **homogeneous** system $x_1 - 3x_2 + 5x_3 = 0$.