Math 1553 Worksheet §1.2, intro to §1.3

Solutions

1. **a)** Circle the 'operations' that are legal to use in row reduction, in other words, the operations that will not change the solution set of an arbitrary linear system.

(1) $R_2 = R_3 + 4R_2$

(2) $R_1 = R_2 - R_3$

(3) $R_2 = R_2 + (R_1)^5$

(4) $R_3 = R_3 - \ln(R_2)$

b) These are row operations only. Try performing a column operation: for example, try doubling any column in $\left(\begin{array}{c|c} 1 & 1 \end{array}\right)$. What happens to the solution set?

Solution.

(a) Only (1) is a legit operation in row reduction.
(2) $R_1 = R_2 - R_3$ is not because it removed the R_1, so we lose the information in R_1.
(3), (4) have nonlinear operations $(R_1)^5$, $\ln R_2$.

(b) The solution set for $\left(\begin{array}{c|c} 1 & 1 \end{array}\right)$ is $x = 1$. Doubling any column in $\left(\begin{array}{c|c} 1 & 1 \end{array}\right)$ either changes the augmented matrix to $\left(\begin{array}{c|c} 2 & 1 \end{array}\right)$ or $\left(\begin{array}{c|c} 1 & 2 \end{array}\right)$ corresponding to (different) solution sets $x = \frac{1}{2}$ and $x = 2$. You cannot perform column operations, as that will change the solution set of the linear system.

2. **a)** Which of the following matrices are in row echelon form (REF)? Which are in reduced row echelon form (RREF)?

b) For the matrices that are in REF or RREF, which entries are the pivots? What are the pivot columns?

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \left(\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 1 \\ 0 & 0 & 4 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

c) How many nonzero entries are there in a pivot column of a matrix in RREF?

Solution.

a) The first is in reduced row echelon form; the second is in row echelon form. The third is neither.

b) The pivots are in red; the other entries in the pivot columns are in blue. The third is not in REF, but with one swap $R_2 \longleftrightarrow R_3$ it will be REF and pivots are easy to find.

c) In a pivot column of RREF, we will have to clear all entries above and below the pivot. This means it has only 1 nonzero entry.

3. Each matrix below is in RREF. In each case, determine whether the corresponding system of linear equations is consistent, and if so, how many solutions does it have?

$$\text{(a)}\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \quad \text{(b)}\ \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 7 \end{array}\right), \quad \text{(c)}\ \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solution.

a) [no solution]. There is a pivot in the rightmost column.

b) [infinitely many solutions]. x_2 is a free variable.

c) [infinitely many solutions]. Every point (x, y, z) in $\mathbb{R}^3$ satisfies $0x + 0y + 0z = 0$.

4. Write the augmented matrix for the corresponding system of equations and put it in reduced row echelon form. How many solutions does the system have?

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ -3x_2 - 6x_3 &= -3. \end{aligned}$$

Solution.

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{array}\right) \xrightarrow{R_2 = R_2 + 4R_1} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2 = R_2 \div 3} \left(\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The system is consistent and has a free variable (the variable x_3 does not have a pivot in its column), so there are infinitely many solutions.

It is possible that your lecture has not gotten to the following yet (if not, you will get there early next Monday): we can solve for x_1 and x_2 in terms of the free variable x_3 to get the "parametric form"

$$x_1 = -2 + 5x_3, \qquad x_2 = 1 - 2x_3, \qquad x_3 = x_3 \quad (x_3 \text{ real}).$$

This consistent system in three variables has one free variable, so the solution set is a line in $\mathbf{R}^3$.