

## Math 1553 Worksheet §5.5-5.6

1. True or false. Justify your answer.

A  $3 \times 3$  matrix  $A$  can have a non-real complex eigenvalue with multiplicity 2.

### Solution.

No. If  $c$  is a (non-real) complex eigenvalue with multiplicity 2, then its conjugate  $\bar{c}$  is an eigenvalue with multiplicity 2 since complex eigenvalues always occur in conjugate pairs. This would mean  $A$  has a characteristic polynomial of degree 4 or more, which is impossible since  $A$  is  $3 \times 3$ .

2. Let  $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ . Find all eigenvalues of  $A$ . For each eigenvalue, find an associated eigenvector.

### Solution.

The characteristic polynomial is

$$\lambda^2 - \text{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 5$$

$$\lambda^2 - 2\lambda + 5 = 0 \iff \lambda = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

For the eigenvalue  $\lambda = 1 - 2i$ , we use the shortcut trick you may have seen in class: the first row  $(a \ b)$  of  $A - \lambda I$  will lead to an eigenvector  $\begin{pmatrix} -b \\ a \end{pmatrix}$  (or equivalently,  $\begin{pmatrix} b \\ -a \end{pmatrix}$  if you prefer).

$$(A - (1 - 2i)I \mid 0) = \left( \begin{array}{cc|c} 2i & 2 & 0 \\ (*) & (*) & 0 \end{array} \right) \implies v = \begin{pmatrix} -2 \\ 2i \end{pmatrix}.$$

From the correspondence between conjugate eigenvalues and their eigenvectors, we know (without doing any additional work!) that for the eigenvalue  $\lambda = 1 + 2i$ , a corresponding eigenvector is  $w = \bar{v} = \begin{pmatrix} -2 \\ -2i \end{pmatrix}$ .

If you used row-reduction for finding eigenvectors, you would find  $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$  as an eigenvector for eigenvalue  $1 - 2i$ , and  $w = \begin{pmatrix} -i \\ 1 \end{pmatrix}$  as an eigenvector for eigenvalue  $1 + 2i$ .

3. A video game offers participants the chance to play as one of two characters: Archer or Barbarian. The game has 100 million players.

In 2023:

Archer is played by 60 million players.

Barbarian is played by 40 million players.

One year later, in 2024:

- 60% of the people who started with the Archer still play with the Archer, while 40% have switched to Barbarian.
- 70% of the customers who started with the Barbarian still play with the Barbarian, while 30% have switched to Archer.

- a) Write down the stochastic matrix  $A$  which represents the change in each character's popularity from 2023 to 2024, and use it to find the number of people who played with each character in 2024.
- b) Suppose the trend continues each year. In the distant future, who will be the most popular character? What will be the player distribution?

**Solution.**

a)

$$A = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}, \quad A \begin{pmatrix} 60 \\ 40 \end{pmatrix} = \begin{pmatrix} 48 \\ 52 \end{pmatrix}.$$

Reminder that the entries in each column of a stochastic matrix sum up to 1.

This means that, in 2024: the archer and barbarian will have 52 million and 48 million players (respectively).

- b) To figure out the most popular character over a long period of time, we need to calculate the steady-state for this stochastic matrix, which spans the 1-eigenspace. Solving for  $(A - I)v = 0$  yields

$$\left( \begin{array}{cc|c} -0.4 & 0.3 & 0 \\ 0.4 & -0.3 & 0 \end{array} \right) \xrightarrow{RREF} \left( \begin{array}{cc|c} 1 & -3/4 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

so the 1-eigenspace is the span of  $\begin{pmatrix} 3/4 \\ 1 \end{pmatrix}$  and the steady-state vector for  $A$  is

$$\begin{pmatrix} \frac{3/4}{1+3/4} \\ \frac{1}{1+3/4} \end{pmatrix} = \begin{pmatrix} 3/7 \\ 4/7 \end{pmatrix}$$

Thus, in the long-term, about  $3/7$  of the players will use the archer, while  $4/7$  of the players will use the barbarian. The player base is 100 million, so eventually the distribution of players will approximately be the following:

$$\text{Archer} : \frac{3}{7}(100) \approx 42.86 \text{ million}$$

$$\text{Barbarian} : \frac{4}{7}(100) \approx 57.14 \text{ million}$$

In the long run, the barbarian will be the most popular character.