Math 1553 Final, SOLUTIONS, Spring '26, Version A

Name		**GT ID**	

Circle your exam room below. If your name or GT ID is not legible, or if you do not circle your exam room, we may deduct points.

Clough 152: Callis, Hao, and Van Why (lectures E, F, M)

Howey L1: Jankowski (lectures A+HP and C)

Howey L4: Poudel (lectures L and S)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 100 points, and you have 170 minutes to complete it. Each problem is worth 10 points.
- Calculators and cell phones are not allowed. Aids of any kind (notes, text, etc.) are not allowed. If you use pen, you must use black ink. Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero. Simplify all fractions and trig functions.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 8:50 PM on Tuesday, May 5.

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1. TRUE or FALSE. Clearly fill in the bubble for your answer. If the statement is *ever* false, fill in the bubble for False. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

   (a) If $\{v_1, v_2, v_3\}$ is linearly dependent set of vectors in $\mathbf{R}^n$, then v_3 must be a linear combination of v_1 and v_2.

   ○ True

   ● False

   (b) Suppose A is a 3×3 matrix. If the equation $Ax = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$ has exactly one solution, then A must be invertible.

   ● True

   ○ False

   (c) If $\lambda = 2$ is an eigenvalue of an $n \times n$ matrix A, then the 2-eigenspace of A must be a subspace of $\mathbf{R}^n$.

   ● True

   ○ False

   (d) Let A be an $n \times n$ matrix whose eigenvalues are real numbers. If A is diagonalizable, then every vector in $\mathbf{R}^n$ can be written as a linear combination of the eigenvectors of A.

   ● True

   ○ False

   (e) If u is a vector in $\mathbf{R}^n$ and u is orthogonal to itself, then u must be the zero vector.

   ● True

   ○ False

Problem 1 Solution.

(a) False, and taken nearly verbatim from #1 on the Studypalooza problems list. For example, take $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $\{v_1, v_2, v_3\}$ is linearly dependent but v_3 is not a linear combination of v_1 and v_2.

(b) True: If A is 3×3 and $Ax = b$ has a unique solution for some b, then the columns of A are linearly independent, so A is invertible by the Invertible Matrix Theorem.

(c) True: the 2-eigenspace is automatically a subspace of $\mathbf{R}^n$ because it is the null space of $A - 2I$.

(d) True: it is just the statement that there is a basis of $\mathbf{R}^n$ consisting of eigenvectors of A, so A is diagonalizable by the Diagonalization Theorem.

(e) True, taken nearly verbatim from the chapter 6 worksheet. If $u \cdot u = 0$ then $0 = u \cdot u = ||u||^2$, so $||u|| = 0$. The only vector with length 0 is the zero vector.

2. On this page, you do not need to show work, and only your answers are graded. Parts (a) through (d) are unrelated.

   (a) (3 pts) Let V be the set of all vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbf{R}^2$ with the property that $x+2y \geq 0$. Which of the subspace properties does V satisfy? Fill in the bubble for all that apply.

   ● V contains the zero vector.

   ● V is closed under addition. In other words, if u and v are vectors in V, then $u+v$ must be in V.

   ○ V is closed under scalar multiplication. In other words, if u is a vector in V and c is a scalar, then cu must be in V.

   (b) (3 points) Let v_1, v_2, v_3, and v_4 be vectors in $\mathbf{R}^5$, and let $W = \text{Span}\{v_1, v_2, v_3, v_4\}$. Which of the following statements must be true? Fill in the bubble for all that apply.

   ○ $\dim(W) = 4$.

   ● If b is a vector in W, then the vector equation $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = b$ must be consistent.

   ● If $\{v_1, v_2, v_3, v_4\}$ is linearly independent, then the homogeneous vector equation $x_1v_1+x_2v_2+x_3v_3+x_4v_4 = 0$ has only the trivial solution $x_1 = x_2 = x_3 = x_4 = 0$.

   (c) (2 points) Suppose A is an 80×60 matrix whose RREF has 45 pivots. Which **one** of the following describes the null space of A?

   ● $\text{Nul}(A)$ is a 15-dimensional subspace of $\mathbf{R}^{60}$.

   ○ $\text{Nul}(A)$ is a 15-dimensional subspace of $\mathbf{R}^{80}$.

   ○ $\text{Nul}(A)$ is a 35-dimensional subspace of $\mathbf{R}^{60}$.

   ○ $\text{Nul}(A)$ is a 35-dimensional subspace of $\mathbf{R}^{80}$.

   (d) (2 points) Suppose $T : \mathbf{R}^3 \to \mathbf{R}^2$ is an **onto** linear transformation with standard matrix A (so $T(x) = Ax$), and suppose $T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

   Which **one** of the following matrices could be A?

   ○ $\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$ ● $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 & -2 & -1 \\ -1 & 2 & 1 \end{pmatrix}$ ○ $\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

Problem 2 Solution.

(a) Note that V is the set of all points in $\mathbf{R}^2$ satisfying $y \geq -\dfrac{x}{2}$. In other words, it is the set of all points (x, y) in $\mathbf{R}^2$ that are on, or above, the line $y = -x/2$.

(i) Yes, V contains the zero vector since $0 + 2(0) \geq 0$.

(ii) Yes, V is closed under addition. We could see this either by graphing the region and observing that the sum of two vectors in V must also be in V, or by using algebra to see that the inequality $y \geq -x/2$ is preserved if we add two points that satisfy it.

(iii) No, V is not closed under scalar multiplication. For example, note that $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is in the region since $1 + 2(0) \geq 0$, but $-v = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ is not in the region since $-1 + 2(0) < 0$.

(b) (i) is not necessarily true: if $\{v_1, v_2, v_3, v_4\}$ is any linearly dependent set of vectors, then $\dim(W) < 4$.

(ii) is true, directly from the definition of span. If b is in W, then b is in the span of those 4 vectors, which means that b is a linear combination of them.

(iii) is true, directly from the definition of linear independence.

(c) There are 45 pivots and 60 columns, so there are exactly $60 - 45 = 15$ columns without pivots. This means that $Ax = 0$ has 15 free variables, so the null space of A is 15-dimensional. The null space of A is a subspace of $\mathbf{R}^{60}$ because A has 60 columns.

(d) The matrix A must be 2×3, its third column must be $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and it must have two pivots since T is onto. The only answer choice that satisfies all of these properties is $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

3. On this page, you do not need to show work. Only your answers are graded. Parts (a) through (e) are unrelated.

(a) (2 points) Suppose that $S : \mathbf{R}^3 \to \mathbf{R}^4$ and $T : \mathbf{R}^2 \to \mathbf{R}^3$ are linear transformations. Which **one** of the following must be true about the composition $S \circ T$?

○ $S \circ T$ is one-to-one.

○ $S \circ T$ cannot be one-to-one.

○ $S \circ T$ is onto.

● $S \circ T$ cannot be onto.

(b) (2 points) Select the **one** matrix below that satisfies $\text{Col}(A) = \text{Span}\left\{\begin{pmatrix}1\\-3\end{pmatrix}\right\}$ and $\text{Nul}(A) = \text{Span}\left\{\begin{pmatrix}4\\1\end{pmatrix}\right\}$.

○ $A = \begin{pmatrix}1 & -3\\4 & 1\end{pmatrix}$ ○ $A = \begin{pmatrix}1 & 4\\-3 & 1\end{pmatrix}$ ○ $A = \begin{pmatrix}1 & 2\\-3 & -6\end{pmatrix}$

● $A = \begin{pmatrix}1 & -4\\-3 & 12\end{pmatrix}$ ○ $A = \begin{pmatrix}1 & -1/4\\-3 & 3/4\end{pmatrix}$ ○ $A = \begin{pmatrix}4 & 4/3\\1 & 1/3\end{pmatrix}$

(c) (2 points) Let $T : \mathbf{R}^3 \to \mathbf{R}^4$ be the linear transformation given by

$$T(x, y, z) = (x - y,\ x + y,\ x + 2y + z,\ z).$$

Which **one** of the following statements is true?

○ T is one-to-one and onto.

● T is one-to-one but not onto.

○ T is onto but not one-to-one.

○ T is neither one-to-one nor onto.

(d) (2 points) Which of the following transformations are linear? Fill in the bubble for all that apply.

● The transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ of orthogonal projection onto the line $y = 8x$.

● The transformation $T : \mathbf{R}^2 \to \mathbf{R}^3$ given by $T(x, y) = (x, x - y, x + y)$

(e) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that first reflects each vector $\begin{pmatrix}x\\y\end{pmatrix}$ across the x-axis, then rotates clockwise by 90°. Find $T\begin{pmatrix}0\\1\end{pmatrix}$.

○ $\begin{pmatrix}1\\0\end{pmatrix}$ ○ $\begin{pmatrix}0\\1\end{pmatrix}$ ● $\begin{pmatrix}-1\\0\end{pmatrix}$ ○ $\begin{pmatrix}0\\-1\end{pmatrix}$ ○ $\begin{pmatrix}-1\\1\end{pmatrix}$

Problem 3 Solution.

(a) If we follow domains and codomains, we see that the composition $S \circ T$ has domain $\mathbf{R}^2$ and codomain $\mathbf{R}^4$, so its standard matrix must be 4×2. Therefore, $S \circ T$ **cannot be onto**, because it is not possible for a 4×2 matrix to have a pivot in every row. Note that $S \circ T$ **might** be one-to-one, but it might not be one-to-one (for example, the matrix could be the 4×2 zero matrix).

(b) Only one choice has null space equal to the span of $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, namely $A = \begin{pmatrix} 1 & -4 \\ -3 & 12 \end{pmatrix}$.

We can also check that this matrix's column space is the span of $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

(c) The matrix for T is $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

If we row-reduce this matrix, we see it has exactly 3 pivots, so it has a pivot in every column but not every row. Therefore, T is one-to-one but not onto.

(d) (i) is linear, since orthogonal projections onto subspaces are always linear transformations.

(ii) is linear, since it is the matrix transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

(e) T reflects $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ across the x-axis to obtain $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, then rotates this vector by 90° clockwise to arrive at $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$.

We could also do it by multiplication. If A is the matrix for the reflection and B is the matrix for the rotation, then we could obtain our answer by taking $BA\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This is not a typo, as we recall that the matrix for reflection goes on the **right** because it represents the **first** operation.

$$BA\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

4. On this page, you do not need to show work. Only your answers are graded. Parts (a) through (e) are unrelated.

(a) (2 points) Suppose $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find $\det \begin{pmatrix} 3d & 3e & 3f \\ d-4a & e-4b & f-4c \\ g & h & i \end{pmatrix}$.

○ -1 ○ 3 ○ -3 ○ 4 ○ 4 ○ 6

○ 1 ● 12 ○ -12 ○ none of these

(b) (2 points) Find all values of a so that $\lambda = 0$ is an eigenvalue of $\begin{pmatrix} a & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & a \end{pmatrix}$.

○ $a = 0$ only ○ $a = -2$ only ○ $a = 2$ only ○ $a = 4$ only

○ $a = -4$ only ● $a = 0$ and $a = -4$ ○ $a = 0$ and $a = 4$

(c) (2 points) Find the eigenvalues of $A = \begin{pmatrix} -5 & 2 \\ -1 & -3 \end{pmatrix}$.

○ 5 and 3 ○ 2 and -3 ○ 4 only ○ -2 and 10

○ $5 \pm 3i$ ○ $-5 \pm 3i$ ○ $4 \pm i$ ● $-4 \pm i$

(d) (2 points) Find the area of the triangle with vertices

$$(1, 5), \qquad (2, 7), \qquad (4, -12).$$

○ $3/2$ ○ $11/2$ ○ $13/2$ ○ 13 ● $23/2$

○ $17/2$ ○ 23 ○ 26 ○ 52 ○ 60

(e) (2 points) Write a matrix A that satisfies $(\text{Nul A})^{\perp} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}$. There is no partial credit, so write your answer carefully.

$$A = \begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$$

Problem 4 Solution.

(a) The matrix is obtained by first switching rows 1 and 2 (new determinant of -1), then multiplying the new first row by 3 and the new second row by -4 (new determinant of $(-1)(3)(-4) = 12$), then does a row replacement by adding 1/3 of the first row to the second which does not change the determinant.

(b) Note that $\lambda = 0$ is an eigenvalue of a matrix if and only if the matrix is not invertible, so we solve $\det(A) = 0$ using the third row cofactor expansion:

$$0 = \det \begin{pmatrix} a & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & a \end{pmatrix} = a(-1)^{3+3} \det \begin{pmatrix} a & 2 \\ 2 & -1 \end{pmatrix} = a(-a-4),$$

so $a = 0$ or $a = -4$.

(c) The characteristic equation is

$$0 = \det(A - \lambda I) = (-5-\lambda)(-3-\lambda) + 2 = \lambda^2 + 8\lambda + 17,$$

so

$$\lambda = \frac{-8 \pm \sqrt{8^2 - 4(17)(1)}}{2} = \frac{-8 \pm \sqrt{64-68}}{2} = \frac{-8 \pm 2i}{2} = -4 \pm i.$$

(d) We can determine the area of the triangle using two sides.

The vector from $(1,5)$ to $(2,7)$ is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

The vector from $(1,5)$ to $(4,-12)$ is $\begin{pmatrix} 3 \\ -17 \end{pmatrix}$.

Therefore, the area of the triangle is

$$\frac{1}{2}\left|\det \begin{pmatrix} 1 & 3 \\ 2 & -17 \end{pmatrix}\right| = \frac{1}{2}\left|-17-6\right| = \frac{23}{2}.$$

(e) $(\text{Nul } A)^{\perp} = \text{Row } A$, so we need a matrix whose row space is the span of $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

This means A must have exactly one pivot, and every row must be a scalar multiple of $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$. Many answers are possible, for example

$$A = \begin{pmatrix} 1 & 1 & -2 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 2 & -4 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{etc.}$$

5. On this page, you do not need to show work, and only your answers are graded. Parts (a) through (d) are unrelated.

(a) (3 points) Let A be a 4×4 matrix with characteristic polynomial

$$\det(A - \lambda I) = \lambda^2(2-\lambda)(4-\lambda).$$

Which of the following statements must be true? Fill in the bubble for all that apply.

● If the null space of A is a plane, then A is diagonalizable.

○ $\det(A) = 8$.

○ $\dim(\text{Col } A) = 2$.

(b) (3 points) Let A be the 2×2 matrix that reflects each vector $\begin{pmatrix} x \\ y \end{pmatrix}$ across the line $y = 5x$. Which of the following must be true? Fill in the bubble for all that apply.

● A is invertible.

○ The vector $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ is an eigenvector of A.

● A is diagonalizable.

(c) (2 points) Consider an internet with four websites A, B, C, and D, satisfying:

- Site A links to B, C, and D.
- Site B links to A and D.
- Site C links to A and B.
- Site D links only to C.

What is the importance matrix for this internet?

● $\begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$ ○ $\begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ ○ $\begin{pmatrix} 1/3 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 1/3 & 0 & 1/2 & 1 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$

○ $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$ ○ $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ ○ $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

(d) (2 points) Let $A = \begin{pmatrix} 0.2 & 0.3 \\ 0.8 & 0.7 \end{pmatrix}$. What is the steady-state vector for A?

○ $\begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}$ ○ $\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$ ○ $\begin{pmatrix} 1/4 \\ 3/4 \end{pmatrix}$ ● $\begin{pmatrix} 3/11 \\ 8/11 \end{pmatrix}$ ○ none of these

Problem 5 Solution.

(a) Note that the eigenvalues of A are $\lambda = 0$, $\lambda = 2$, and $\lambda = 4$, where $\lambda = 0$ has algebraic multiplicity 2 and the other eigenvalues have algebraic multiplicity 1 (so automatically geometric multiplicity 1).

(i) is true: if the null space of A is a plane, then the geometric multiplicity of $\lambda = 0$ is 2, so the sum of geometric multiplicities is 4 and A is diagonalizable.

(ii) is not true: $\det(A) = \det(A - 0I) = 0^2(2)(4) = 0$.

(iii) is not necessarily true: if $\lambda = 0$ only has geometric multiplicity 1, then the null space of A is one-dimensional, in which case the rank of A would be 3.
For example, $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(b) (i) is true: the eigenvalues of A are -1 and 1, so A is invertible.

(ii) is not true: the 1-eigenspace is spanned by $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ and the (-1)-eigenspace is spanned by $\begin{pmatrix} 5 \\ -1 \end{pmatrix}$, so $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ is not an eigenvector.

(iii) is true: A is a 2×2 matrix with 2 different real eigenvalues, so it is automatically diagonalizable.

(c) Only two choices are stochastic matrices, so we can eliminate almost all choices immediately. Of the two matrices that are stochastic, the one that matches the websites' links is: $\begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1 \\ 1/3 & 1/2 & 0 & 0 \end{pmatrix}$.

(d) $(A - I \mid 0) = \left(\begin{array}{cc|c} -0.8 & 0.3 & 0 \\ 0.8 & -0.3 & 0 \end{array}\right)$, which gives us $0.8x = 0.3y$, so $x = 3y/8$. This gives us $\begin{pmatrix} 3/8 \\ 1 \end{pmatrix}$ as a 1-eigenvector, so our steady-state vector is

$$\frac{1}{\frac{3}{8}+1}\begin{pmatrix} 3/8 \\ 1 \end{pmatrix} = \frac{1}{11/8}\begin{pmatrix} 3/8 \\ 1 \end{pmatrix} = \frac{8}{11}\begin{pmatrix} 3/8 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/11 \\ 8/11 \end{pmatrix}.$$

This problem was doable with almost no work: just compute to find the answer choice that satisfies $Aw = w$, since the sum of each choice's entries is 1.

6. On this page, you do not need to show work. Only your answers are graded. Parts (a) through (d) are unrelated.

   (a) (2 pts) Let W be the subspace of $\mathbf{R}^3$ consisting of all $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfying $x-3y-z=0$, and let B be the matrix for orthogonal projection onto W. Which **one** of the following statements is true?

   ○ $\mathrm{Nul}(B) = W$.

   ○ The rank of B is 1.

   ● The vector $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ is in the column space of B.

   ○ The eigenvalues of B are -1 and 1.

   (b) (2 points) Let v and w be orthogonal vectors satisfying $||v|| = 1$ and $||w|| = 2$. Find the dot product $(v - 8w) \cdot w$.

   ○ 2 ○ 8 ○ -14 ○ -16 ○ -30 ● -32

   (c) (2 points) Let A be an $m \times n$ matrix. Which **one** of the following must be true?

   ○ $(\mathrm{Row} A)^{\perp} = \mathrm{Nul}(A^T)$

   ○ $\dim(\mathrm{Row}\ A) + \dim((\mathrm{Row}\ A)^{\perp}) = m$

   ○ $\mathrm{Row}(A) = \mathrm{Col}(A)$.

   ● $(\mathrm{Col}\ A)^{\perp} = \mathrm{Nul}(A^T)$

   (d) (4 points) Suppose W is a subspace of $\mathbf{R}^3$ and that x is a vector in $\mathbf{R}^3$ whose orthogonal decomposition with respect to W is $x = x_W + x_{W^{\perp}}$, where

$$x_W = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \qquad x_{W^{\perp}} = \begin{pmatrix} 1 \\ 7 \\ 4 \end{pmatrix}.$$

   (i) What is the closest vector to x in W?

   ● $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix}$ ○ $\begin{pmatrix} 2 \\ -8 \\ -3 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 7 \\ 4 \end{pmatrix}$ ○ $\begin{pmatrix} 4 \\ 6 \\ 0 \end{pmatrix}$

   (ii) What is the distance from x to W?

   ○ 3 ○ $\sqrt{11}$ ○ 11 ○ 12 ● $\sqrt{66}$

   ○ 66 ○ $\sqrt{77}$ ○ 77 ○ $\sqrt{79}$ ○ $\sqrt{12}$

Problem 6 Solution.

(a) (i) is false: since B is the matrix for projection onto W we have $\text{Col}(B) = W$, not $\text{Nul}(B) = W$.

(ii) is false: the rank of B is $\dim(W)$, which is 2.

(iii) is true: recall that $\text{Col}(B) = W$, and we observe that $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ is in W since it satisfies $x - 3y - z = 0$, as we check $3 - 3(1) - 0 = 0$.

(iv) is false: the eigenvalues of B are 0 and 1.

(b) Using properties of the dot product, we compute

$$(v - 8w) \cdot w \;=\; v \cdot w - 8w \cdot w \;=\; 0 - 8||w||^2 = -8(4) = -32.$$

(c) (i) is not true, since the correct statement is $(\text{Row } A)^{\perp} = \text{Nul}(A)$.

(ii) is not true: $\text{Row}(A)$ is a subspace of $\mathbf{R}^n$, so the sum in question is equal to n.

(iii) is not true. In fact the row space and column space of A don't even live in the same place unless $m = n$. Even if $m = n$, it is usually not true that $\text{Row}(A) = \text{Col}(A)$. It is the **dimensions** of $\text{Row}(A)$ and $\text{Col}(A)$ that are the same.

(iv) is true, it is a standard fact from section 6.2.

(d) (i) The closest vector to x in W is x_W. We were told $x_W = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$.

(ii) The distance from x to W is $||x_{W^\perp}||$, and we compute

$$||x_{W^\perp}|| = \sqrt{1^2 + 7^2 + 4^2} = \sqrt{66}.$$

7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates vectors by 90° counterclockwise, and let $U : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation $U(x, y, z) = (z - x,\ x - 3y + 4z)$.

(a) Write the standard matrix A for T. Evaluate any trigonometric functions you write. Do not leave your answer in terms of sine and cosine.

$$A = \begin{pmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(b) Write the standard matrix B for U.

Solution: $B = \big(U(e_1) \quad U(e_2) \quad U(e_3)\big) = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -3 & 4 \end{pmatrix}$

(c) Which composition makes sense: $T \circ U$ or $U \circ T$? Fill in the correct bubble below. You do not need to show your work on this part.

● $T \circ U$ ○ $U \circ T$

(d) Compute the standard matrix C for the composition you selected in (c). Put your answer in the space provided below.
Solution: The matrix is

$$C = AB = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & -3 & 4 \end{pmatrix} = \begin{pmatrix} -1 & 3 & -4 \\ -1 & 0 & 1 \end{pmatrix}.$$

8. Free response. Show your work unless otherwise indicated! A correct answer without appropriate work will receive little or no credit. If you need extra space for your work, please use the last page of the exam and indicate this clearly.

For this page, let $A = \begin{pmatrix} 8 & 4 & 16 \\ 2 & 6 & 8 \\ 0 & 0 & 4 \end{pmatrix}$. The only eigenvalues of A are $\lambda = 4$ and $\lambda = 10$.

(a) (6 points) Find a basis for each of the two eigenspaces of A. Enter your answers in the boxes marked below. Write your answers carefully.

Basis for 4-eigenspace: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix} \right\}$

Basis for 10-eigenspace: $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$

Solution:

$$\underline{\lambda = 4} : A - 4I = \begin{pmatrix} 4 & 4 & 16 \\ 2 & 2 & 8 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow[\text{then } R_2 = R_2 - 2R_1]{R_1 = R_1/4} \begin{pmatrix} 1 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{so } (A - 4I \mid 0) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

We get $x_1 + x_2 + 4x_3 = 0$, so $x_1 = -x_2 - 4x_3$ where x_2 and x_3 are free, and $(x_1, x_2, x_3) = (-x_2 - 4x_3, x_2, x_3) = x_2(-1, 1, 0) + x_3(-4, 0, 1)$.

$$\underline{\lambda = 10} : A - 10I = \begin{pmatrix} -2 & 4 & 16 \\ 2 & -4 & 8 \\ 0 & 0 & -6 \end{pmatrix} \to \begin{pmatrix} -2 & 4 & 16 \\ 0 & 0 & 24 \\ 0 & 0 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & -8 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ so } (A - 10I \mid 0) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

This gives us $x_1 - 2x_2 = 0$ and $x_3 = 0$ where x_2 is free, so $x_1 = 2x_2$ and $(x_1, x_2, x_3) = (2x_2, x_2, 0) = x_2(2, 1, 0)$

(b) (3 points) A is diagonalizable. Write an invertible 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$. You do not need to show your work on this part.

Solution: Many answers possible. We need C to be a matrix of linearly independent eigenvectors, and D is the corresponding diagonal matrix of eigenvalues.

$$C = \begin{pmatrix} -1 & -4 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 10 \end{pmatrix}, \quad \text{or}$$

$$C = \begin{pmatrix} 2 & -1 & -4 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad \text{etc.}$$

(c) (1 pt) How many solutions are there to the matrix equation $(A + 4I)x = 0$? You do not need to show your work on this part, and there is no partial credit.

○ No solutions ● Exactly one solution ○ Infinitely many solutions

Solution: $(A + 4I)x = 0$ means $Ax + 4x = 0$, so $Ax = -4x$. We were told that the only eigenvalues of A are $\lambda = 4$ and $\lambda = 10$, so $\lambda = -4$ is NOT an eigenvalue. Therefore, the equation $(A + 4I)x = 0$ has only the trivial solution.

9. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work will receive little or no credit. Parts (a) and (b) are unrelated.

(a) (6 points) Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\}$, and let $x = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$.

Find x_W. In other words, find the orthogonal projection of x onto W. Enter your answer in the space provided below.

$$x_W = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

Solution: With $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ -1 & -2 \end{pmatrix}$, we solve $A^T A v = A^T x$ for v.

$$A^T A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix}, \quad A^T x = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

We row-reduce $\left(\begin{array}{cc|c} 3 & 1 & -2 \\ 1 & 5 & 4 \end{array} \right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array} \right)$, so $v = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and

$$x_W = Av = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

(b) (4 points) Let $W = \text{Span}\left\{ \begin{pmatrix} 5 \\ -3 \end{pmatrix} \right\}$. Find the matrix B for orthogonal projection onto W. In other words, find the matrix B so that $Bx = x_W$ for all x in $\mathbf{R}^2$. Enter your answer in the space provided below.

Solution: With $u = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$, we get

$$B = \frac{1}{u \cdot u} u u^T = \frac{1}{5^2 + (-3)^2} \begin{pmatrix} 5 \\ -3 \end{pmatrix} \begin{pmatrix} 5 & -3 \end{pmatrix} = \frac{1}{34} \begin{pmatrix} 25 & -15 \\ -15 & 9 \end{pmatrix}.$$

10. Free response. Show your work! A correct answer without sufficient work may receive little or no credit.

In this problem, we use the usual convention of (x, y) to denote points in $\mathbf{R}^2$.

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(-2, 10), \quad (1, -5), \quad (4, -8).$$

Enter your answer below:

$$y = -3x + 2.$$

You **must** show appropriate work using least squares. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

Ignore the work for now, just look at the top answer.

No line goes through all three points. The corresponding (inconsistent) system is

$$\begin{aligned} 10 &= M(-2) + B \\ -5 &= M(1) + B \\ -8 &= M(4) + B. \end{aligned}$$

The corresponding matrix equation is $Ax = b$ where $A = \begin{pmatrix} -2 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 10 \\ -5 \\ -8 \end{pmatrix}$.

We solve $A^T A \hat{x} = A^T b$.

$$A^T A = \begin{pmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 21 & 3 \\ 3 & 3 \end{pmatrix} \qquad A^T b = \begin{pmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ -5 \\ -8 \end{pmatrix} = \begin{pmatrix} -57 \\ -3 \end{pmatrix}$$

$$\left(A^T A \mid A^T b \right) = \left(\begin{array}{cc|c} 21 & 3 & -57 \\ 3 & 3 & -3 \end{array} \right) \xrightarrow[\text{then } R_1 = \frac{R_1}{3}]{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 21 & 3 & -57 \end{array} \right) \xrightarrow{R_2 = R_2 - 21R_1} \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -18 & -36 \end{array} \right)$$

$$\xrightarrow{R_2 = -R_2/18} \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & 2 \end{array} \right) \xrightarrow{R_1 = R_1 - R_2} \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array} \right).$$

Thus $\hat{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$. The line is

$$y = -3x + 2.$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.